Dynamic Graph Algorithms

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Outline

Dynamic Graph Problems – Quick Intro

Lecture 1. (Undirected Graphs)
  Dynamic Connectivity

Lecture 2. (Undirected/Directed Graphs)
  Dynamic Shortest Paths

Lecture 3. (Non-dynamic?)
  2-Connectivity in Directed Graphs

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  2-Connectivity in Directed Graphs

Today's Outline

1. 2-Connectivity on directed graphs
2. Algorithms for strong articulation points and strong bridges
3. Experiments
4. Open Problems

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1. **2-Connectivity on directed graphs**
2. Algorithms for strong articulation points and strong bridges
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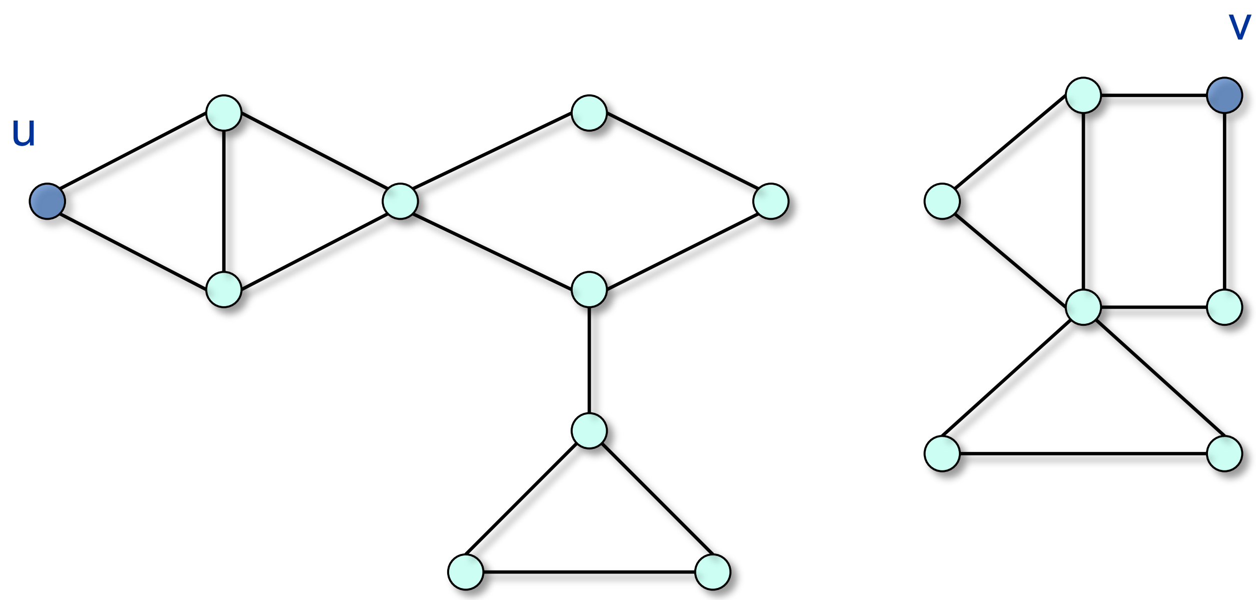
Graph Connectivity

- Fundamental concept in Graph Theory.

- Numerous practical applications, e.g. :
  → Reliable and secure communication
  → Routing
  → Navigation
  → …



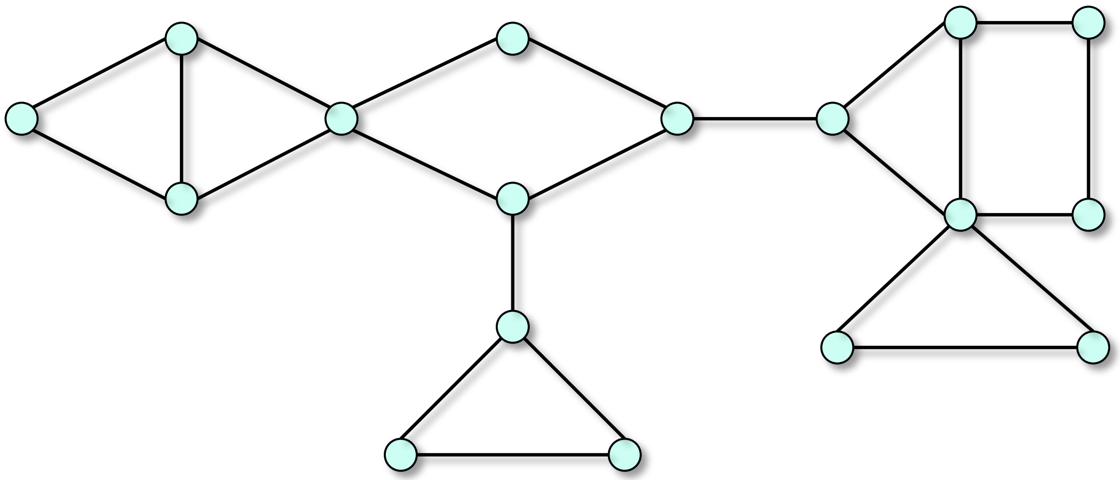
2-Edge Connectivity

Let $G = (V,E)$ be an **undirected** connected graph, with m edges and n vertices.

An edge $e \in E$ is a **bridge** if its removal increases the number of connected components of G.

Graph G is **2-edge-connected** if it has no bridges.

The **2-edge-connected components** of G are its maximal 2-edge-connected subgraphs.



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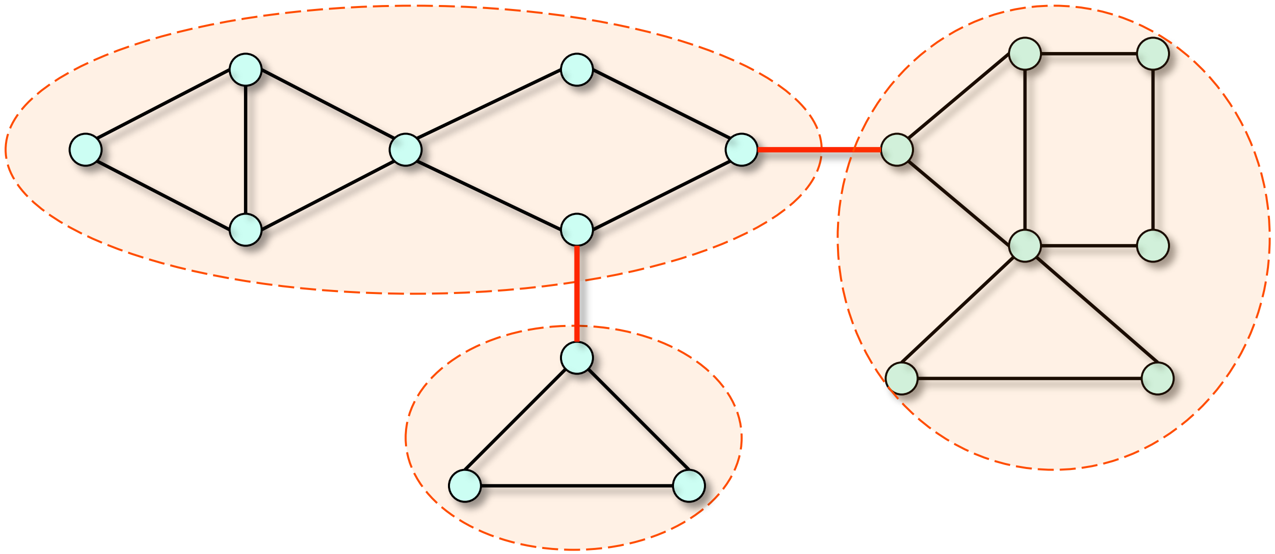
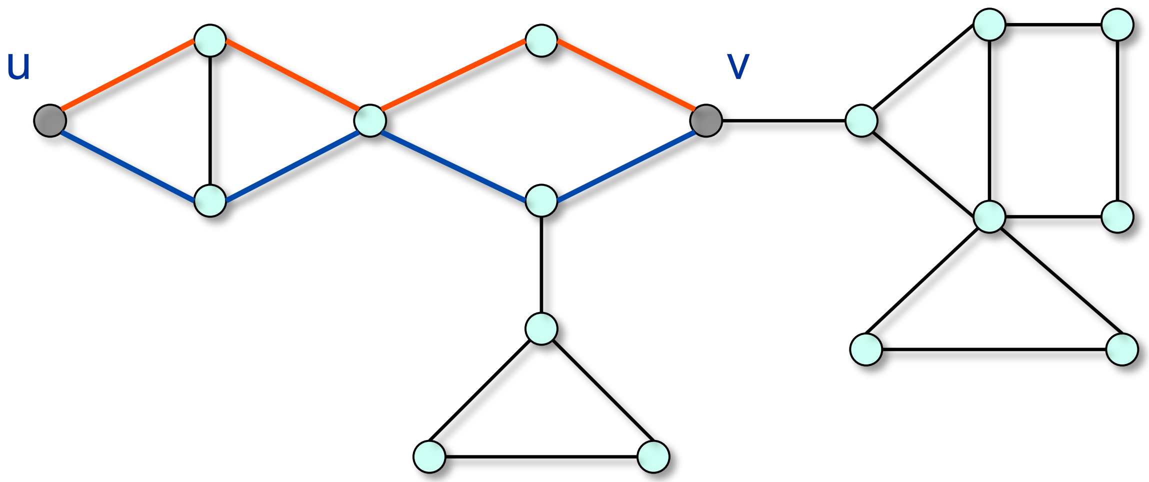


(Point2Point) 2-Edge Connectivity

Vertices u and v are **2-edge-connected** if if there are two edge-disjoint paths between u and v

By Menger's Theorem, vertices u and v are 2-edge-connected if and only removal of any edge leaves them in same connected component.

Can define a **2-edge-connected block** of G as a maximal subset $B \subseteq V$ s. t. u and v are 2-edge-connected for all $u, v \in B$.



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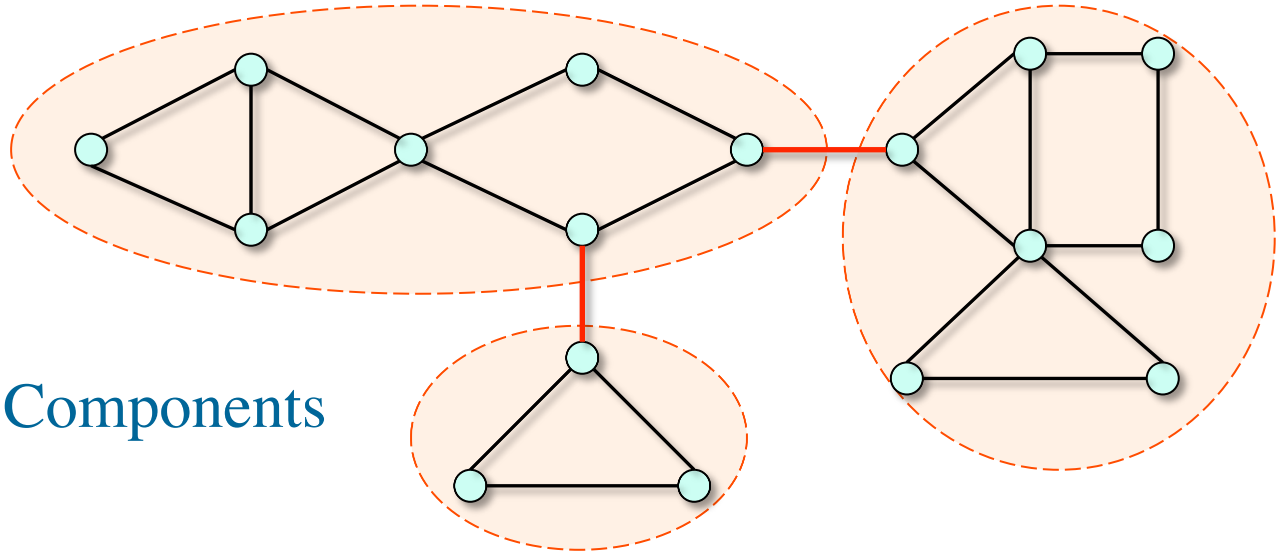

Blocks = Components

2-Vertex Connectivity

Let $G = (V,E)$ be an **undirected** connected graph, with m edges and n vertices.

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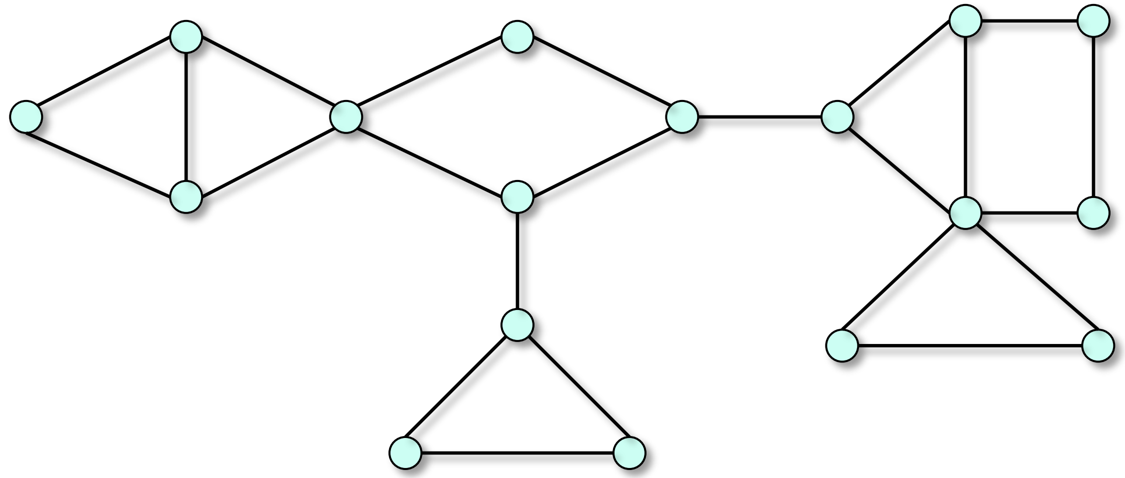

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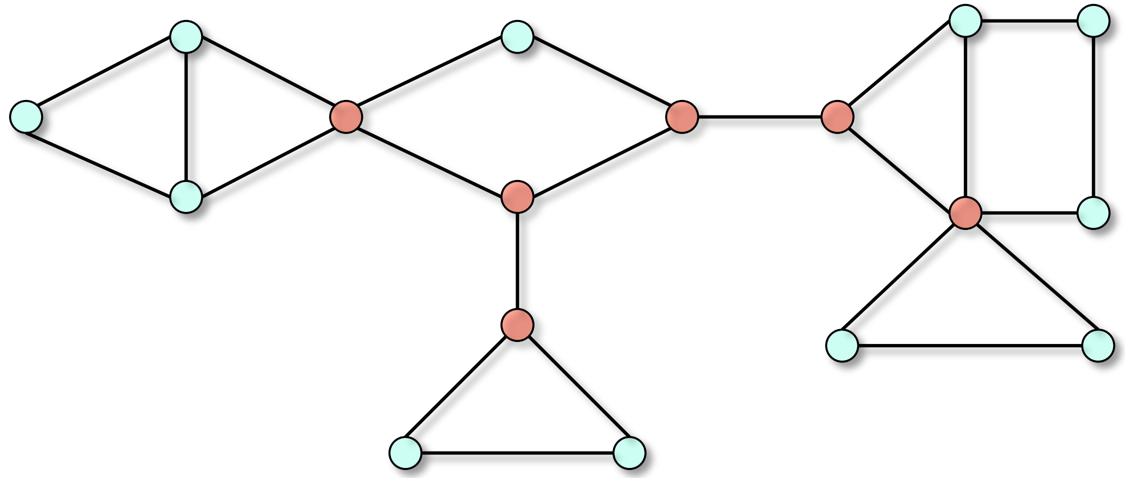

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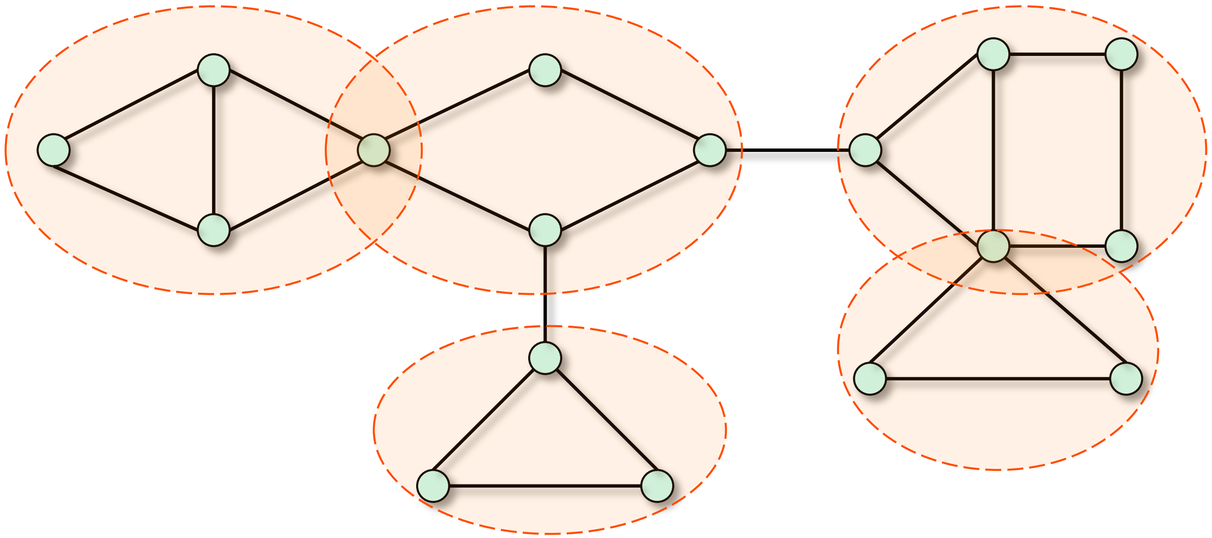

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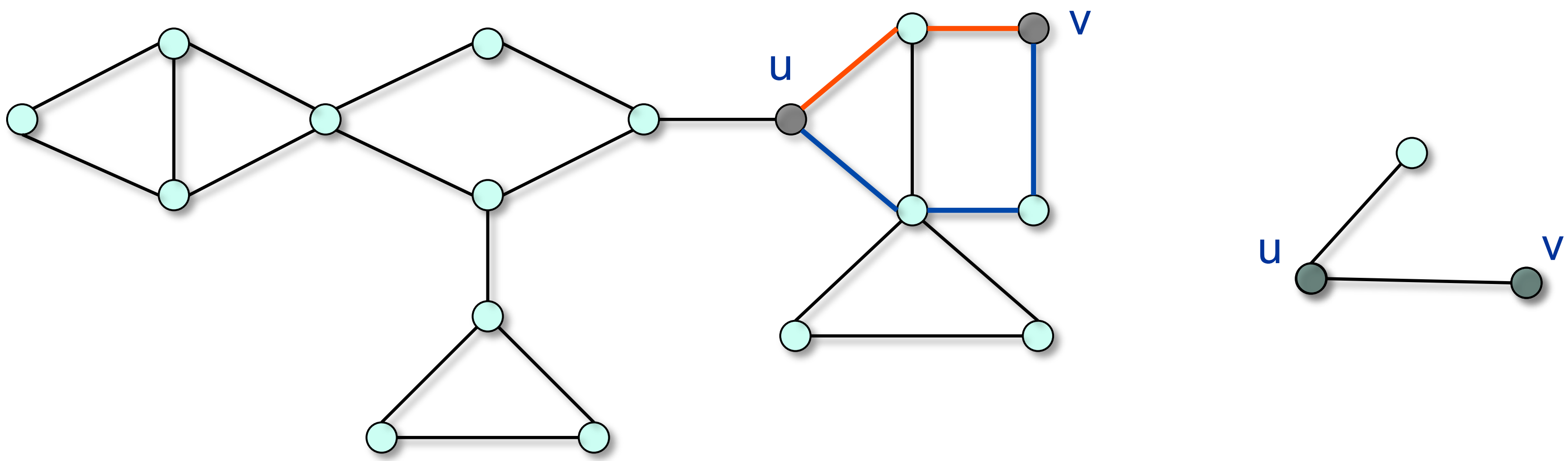

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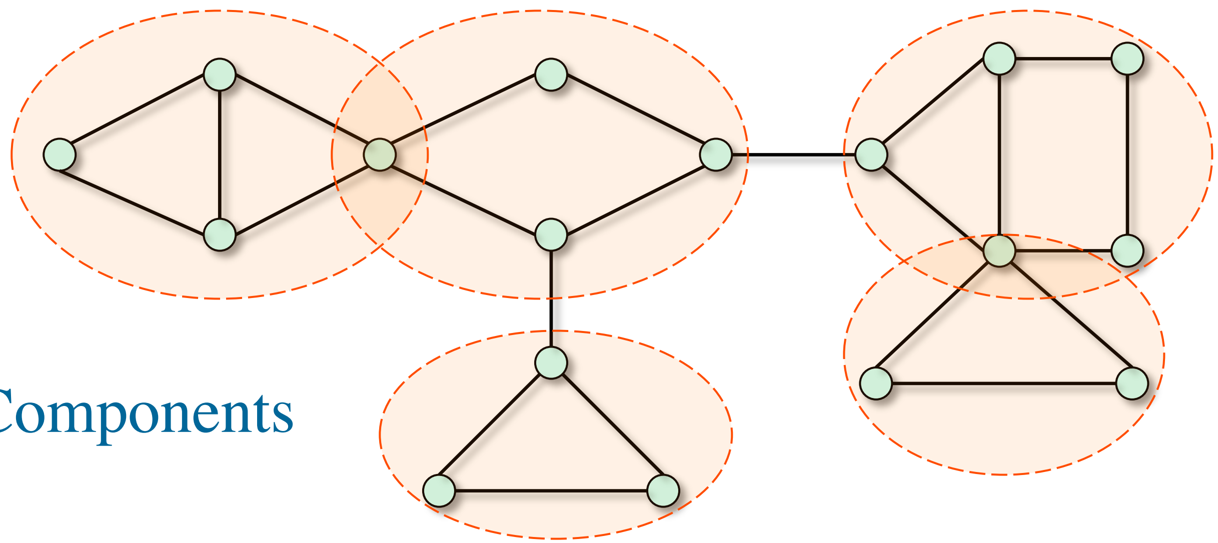


Blocks = Components

Bounds for Undirected *G*

Q1: Find whether *G* is 2-vertex-connected (2-edge-connected).
I.e., find **one** connectivity cut (if any) — ***O(m+n)***

Q2: Find **all** connectivity cuts (articulation points, bridges) in *G* — ***O(m+n)***

Q3: Find the **2-connectivity** (2-vertex-, 2-edge- connected) **blocks** of *G* — ***O(m+n)***

Q4: Find the **2-connectivity** (2-vertex-, 2-edge-connected) **components** of *G* — ***O(m+n)***

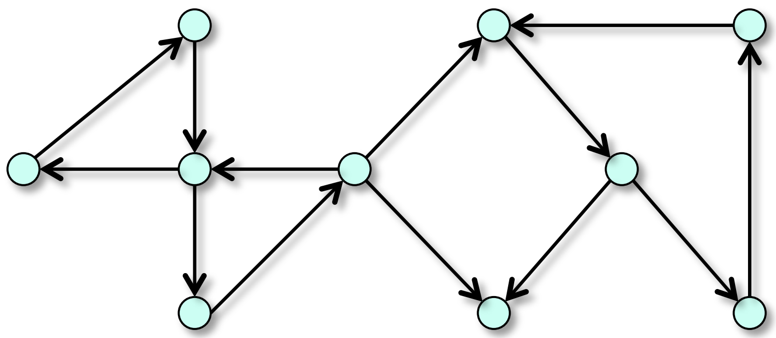
[R.E.Tarjan, SIAM Journal on Computing 1972]

Directed Graphs

Let $G = (V,E)$ be a **directed** graph, with m edges and n vertices.

G is **strongly connected** if there is a directed path from each vertex to every other vertex in G.

The **strongly connected components** (SCCs) of G are its maximal **strongly** connected subgraphs.



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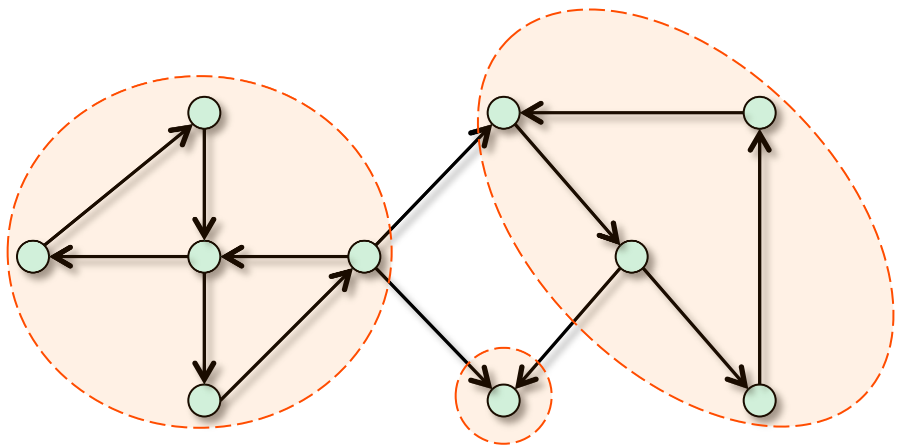

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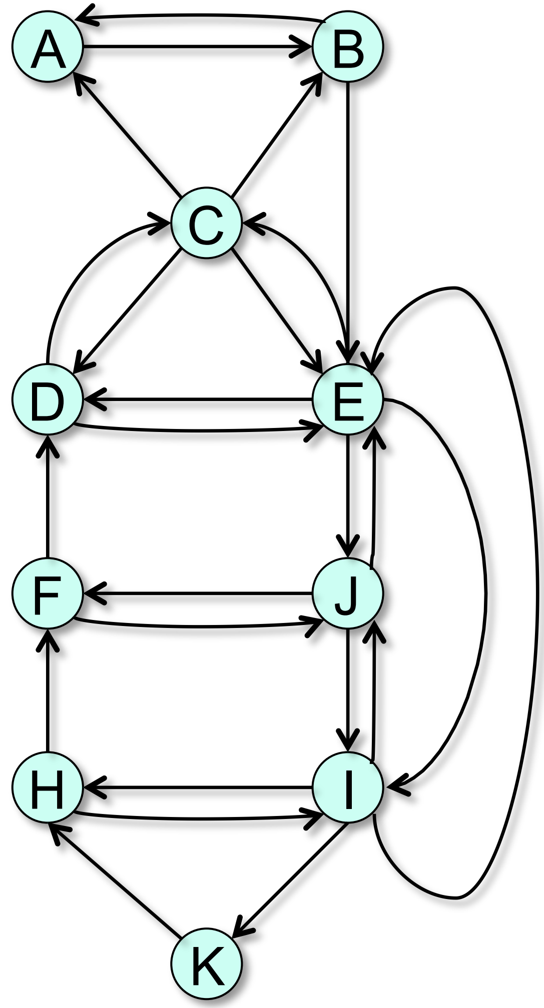

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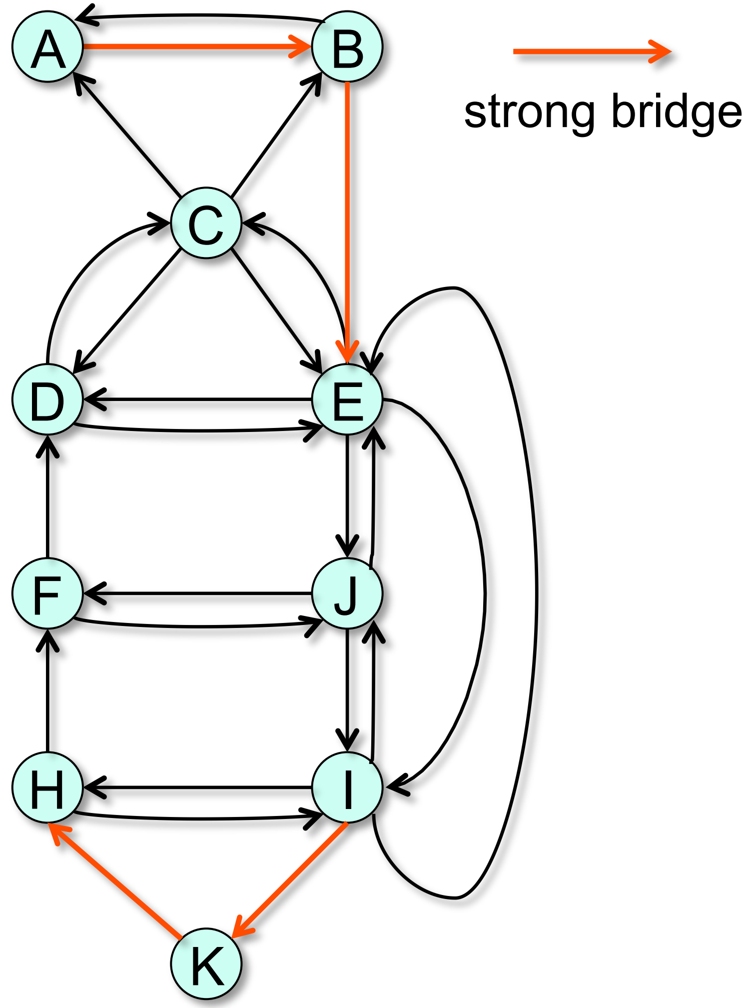

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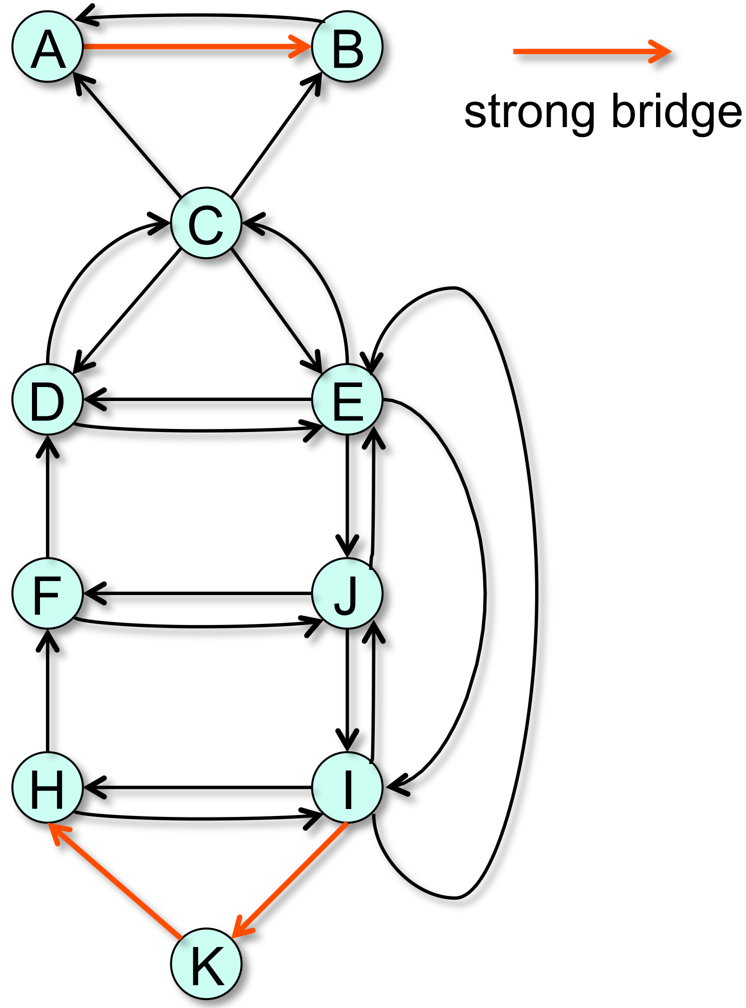

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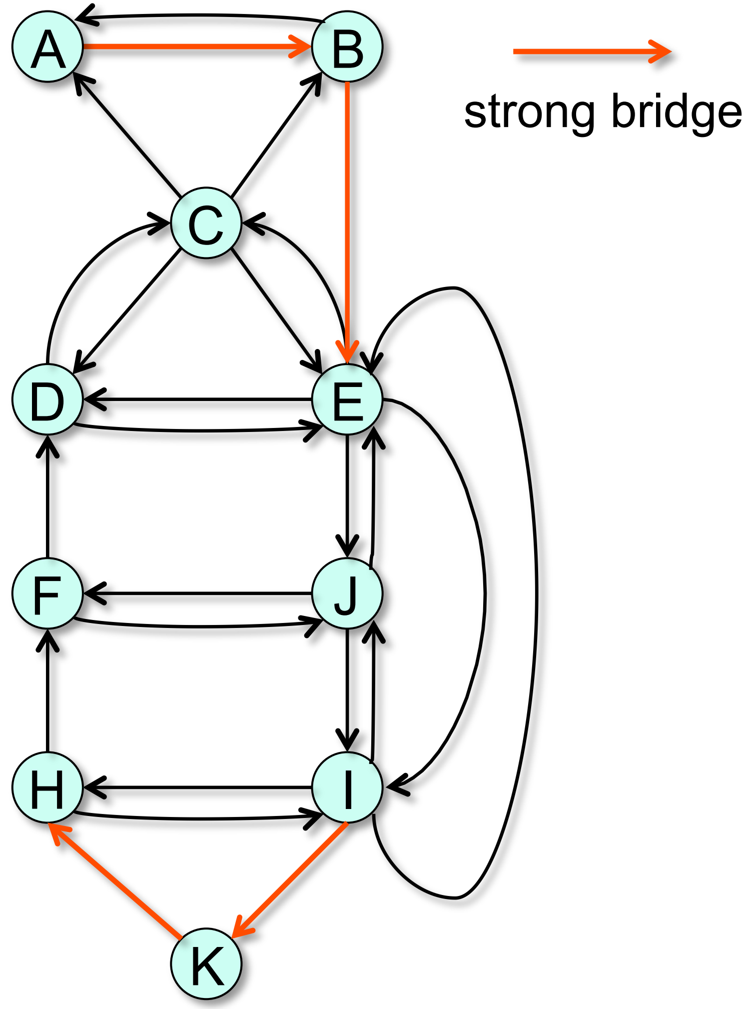

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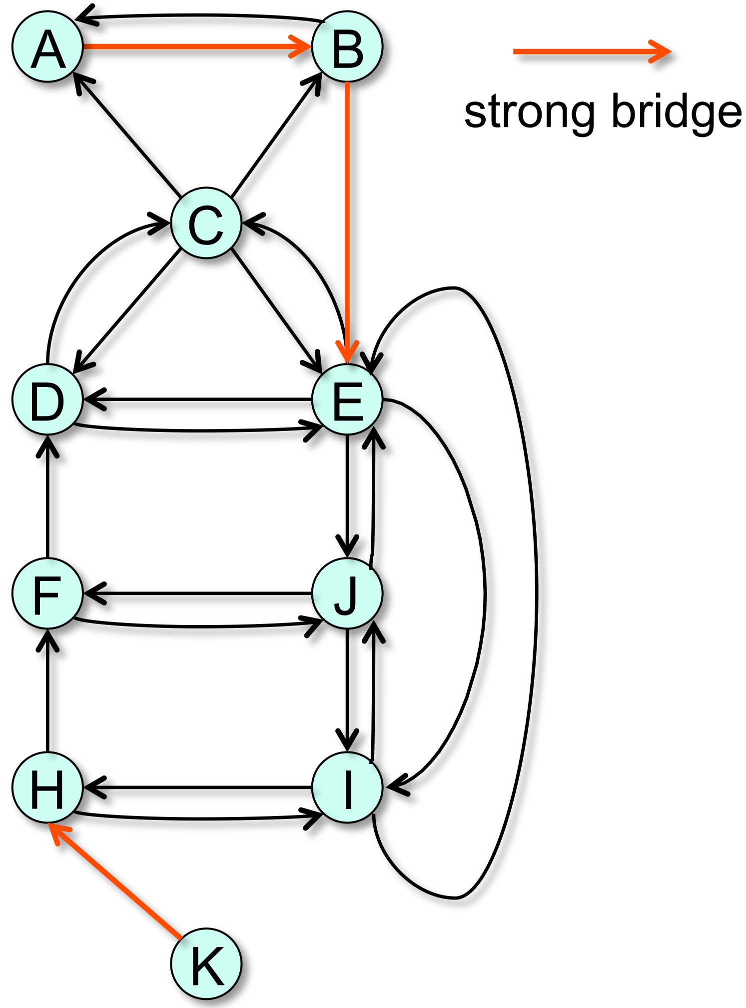

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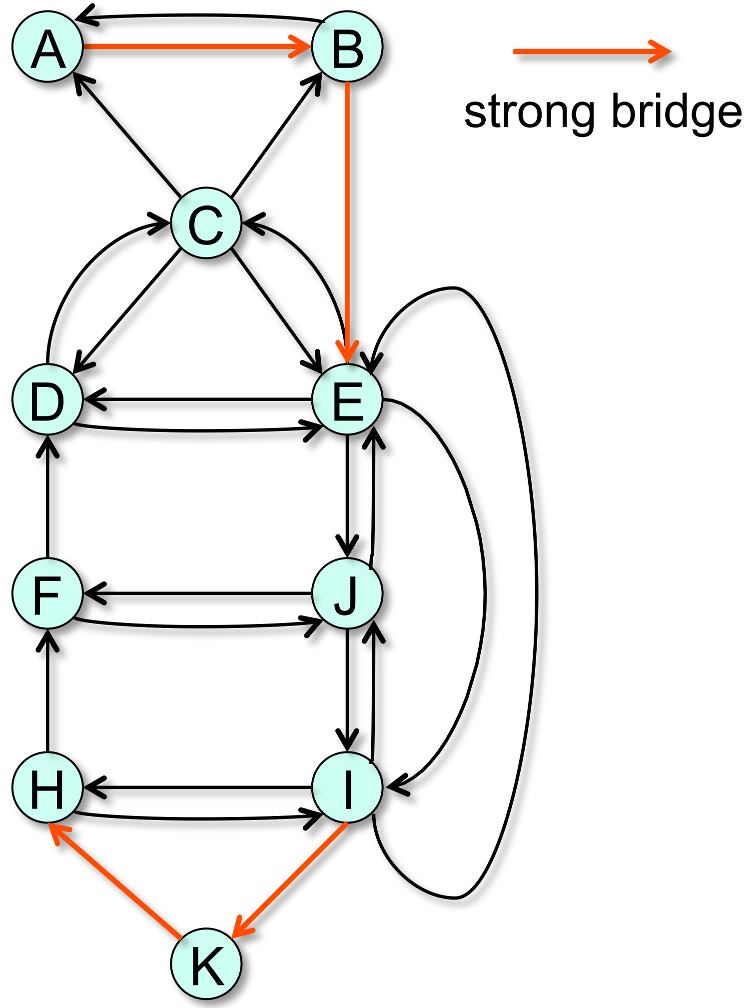

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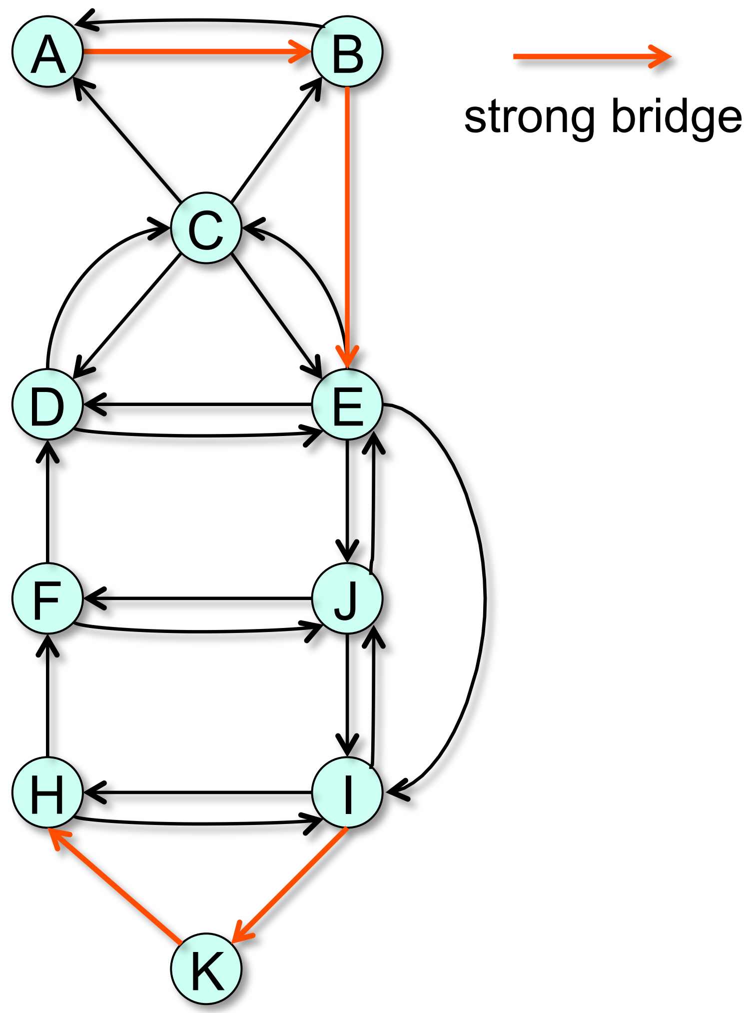

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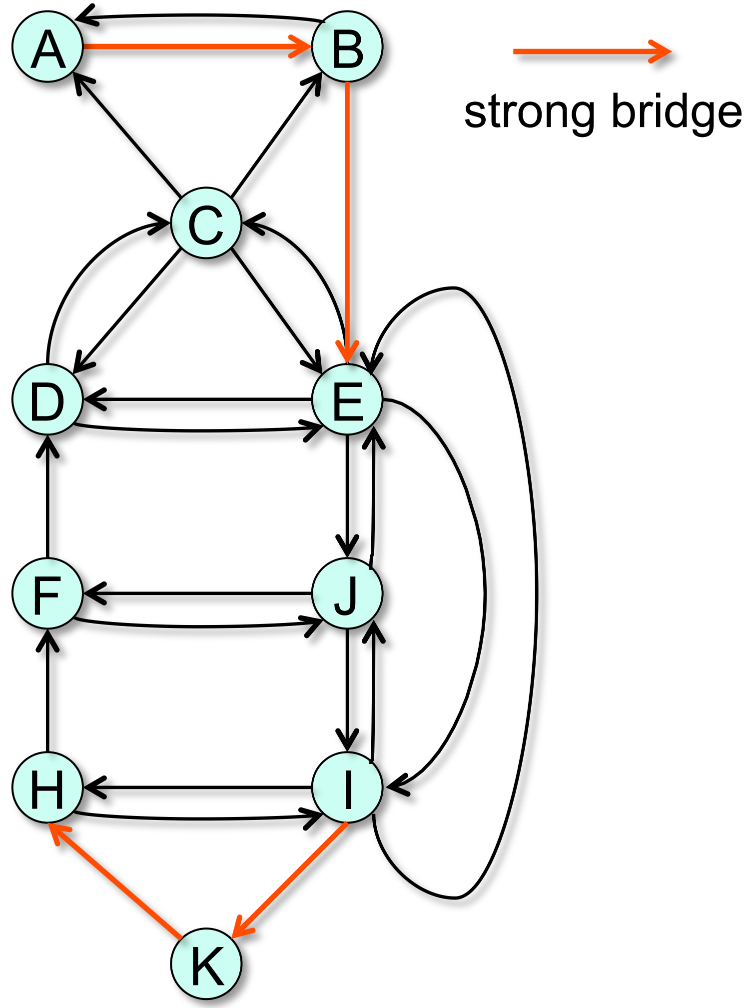

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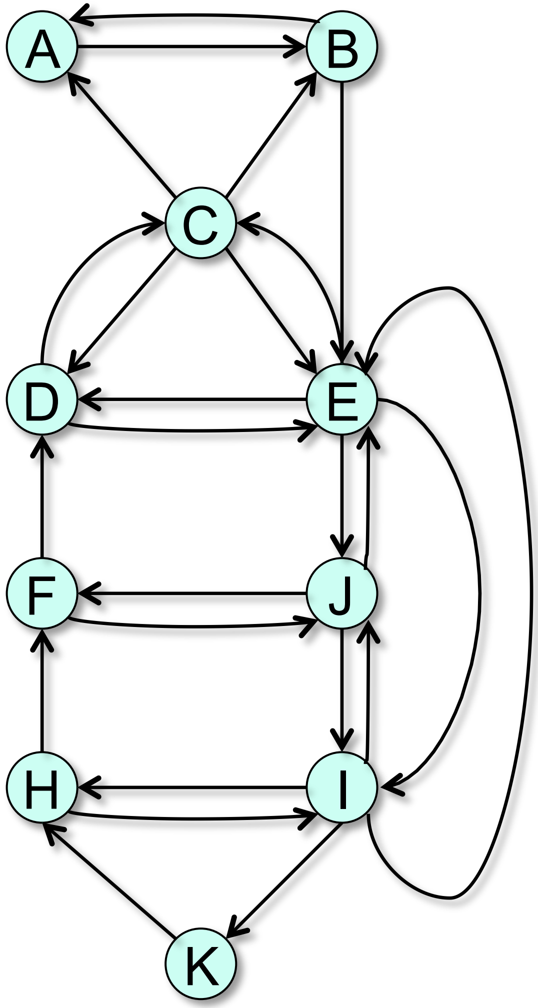
2-Edge Connected Components



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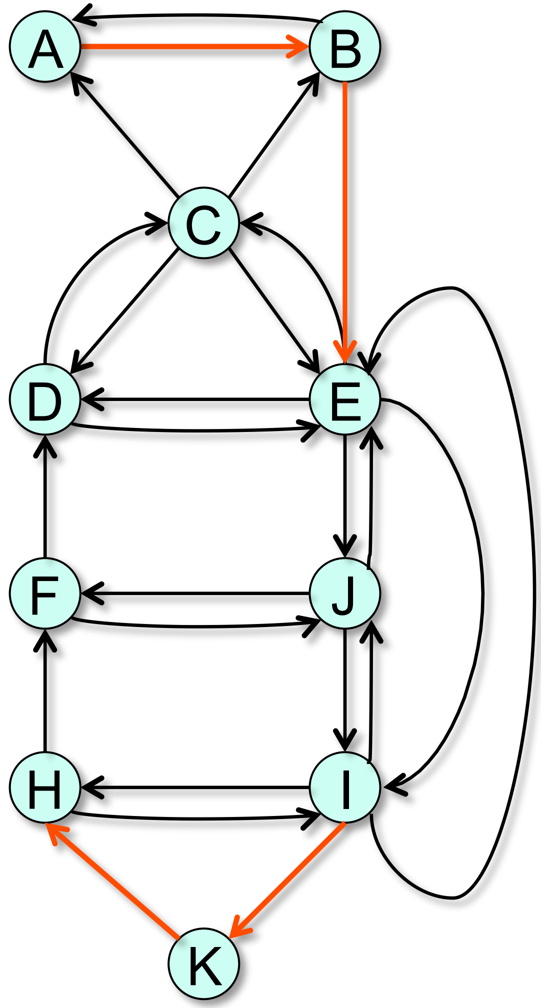

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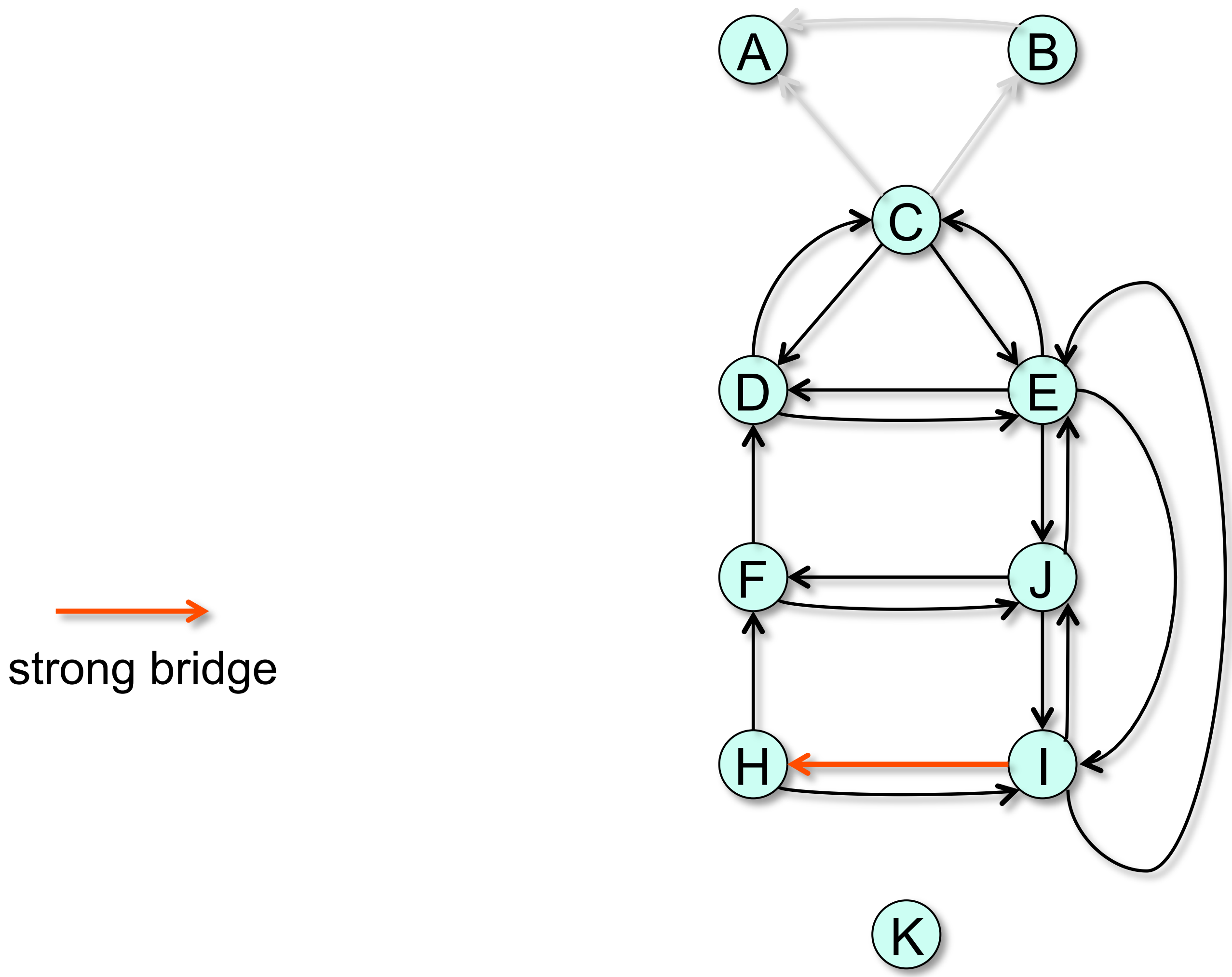

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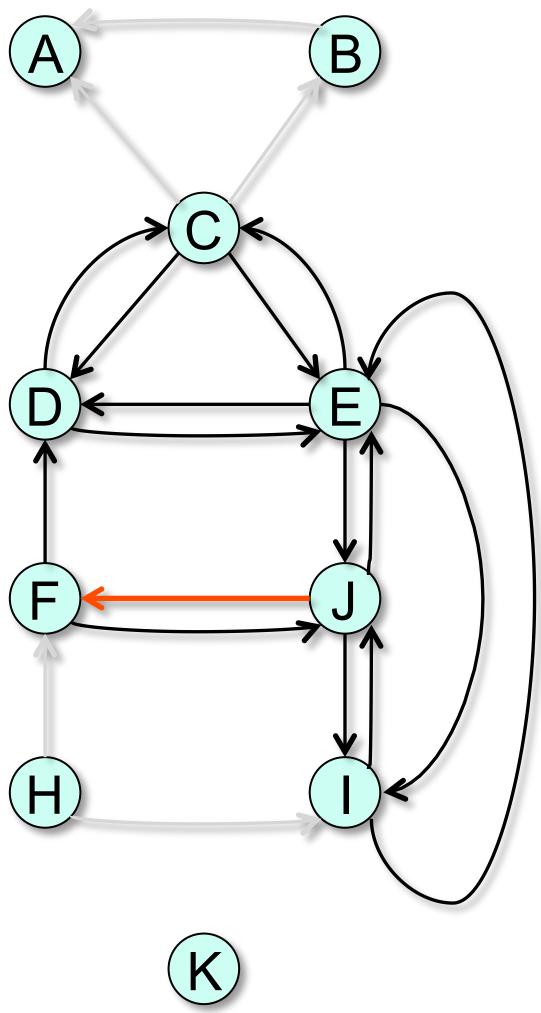

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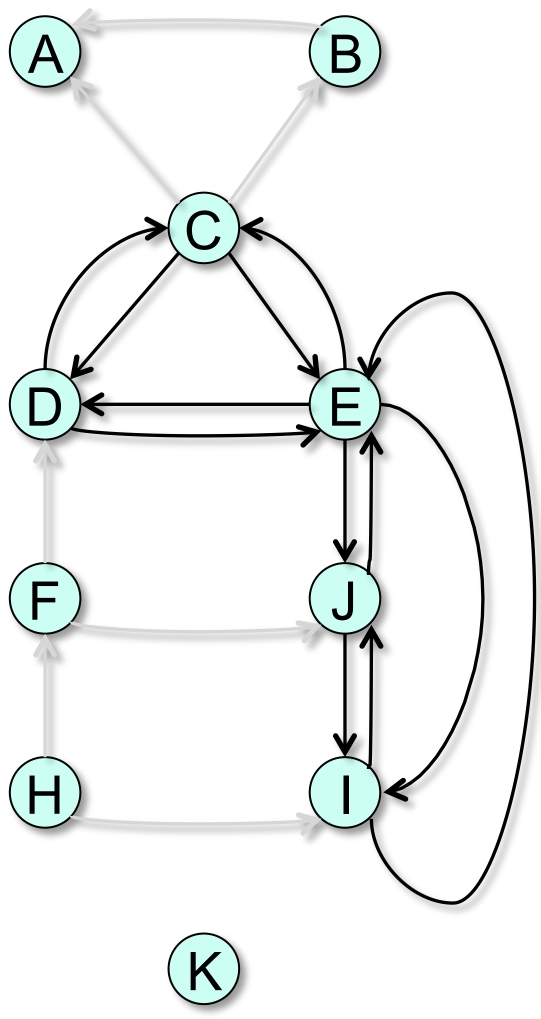

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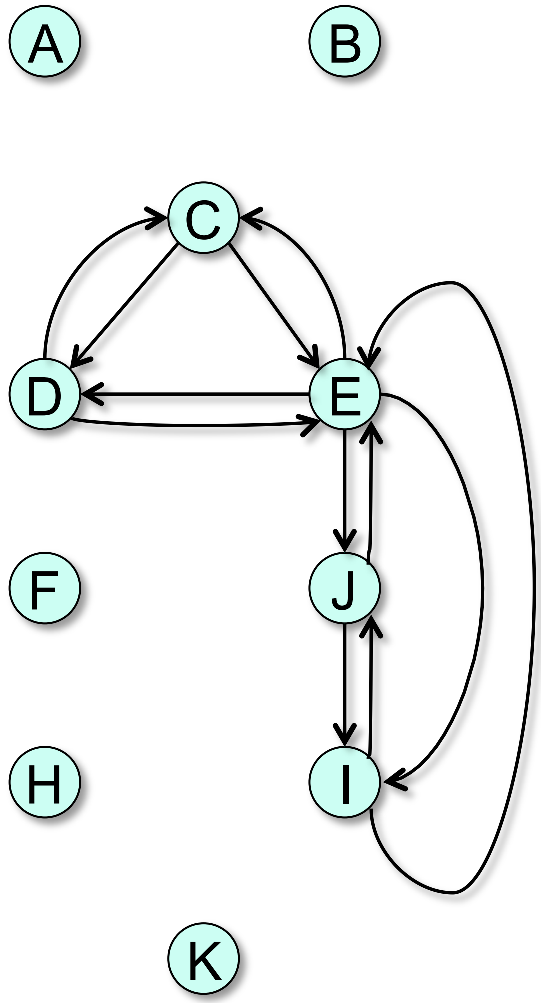

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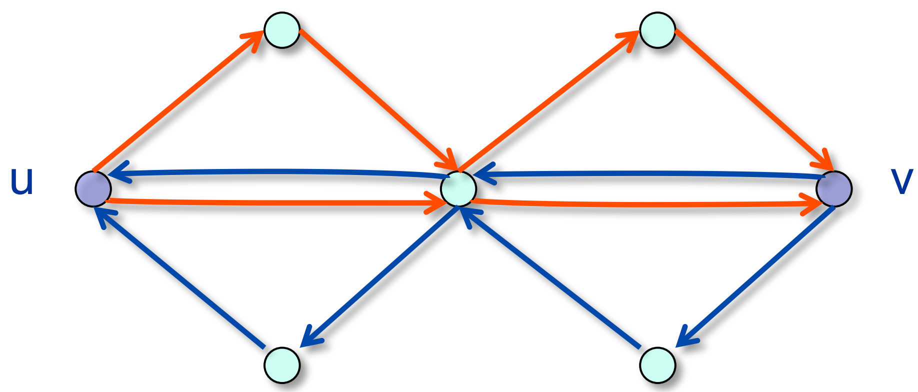
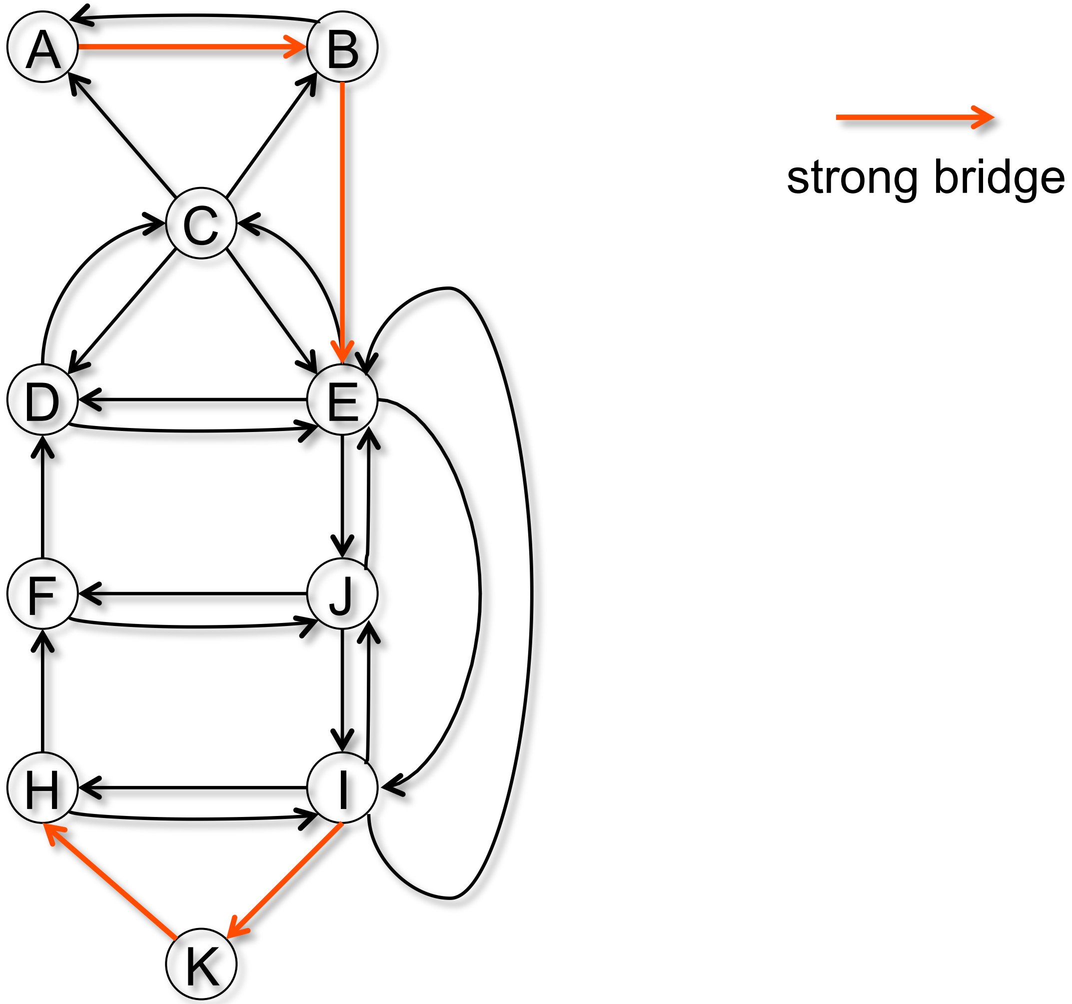


Can define a **2-edge-connected block** of G as a maximal subset $B \subseteq V$ s. t. u and v are **2-edge-connected** for all $u, v \in B$.

2-Edge-Connected Comp's ≠ Blocks



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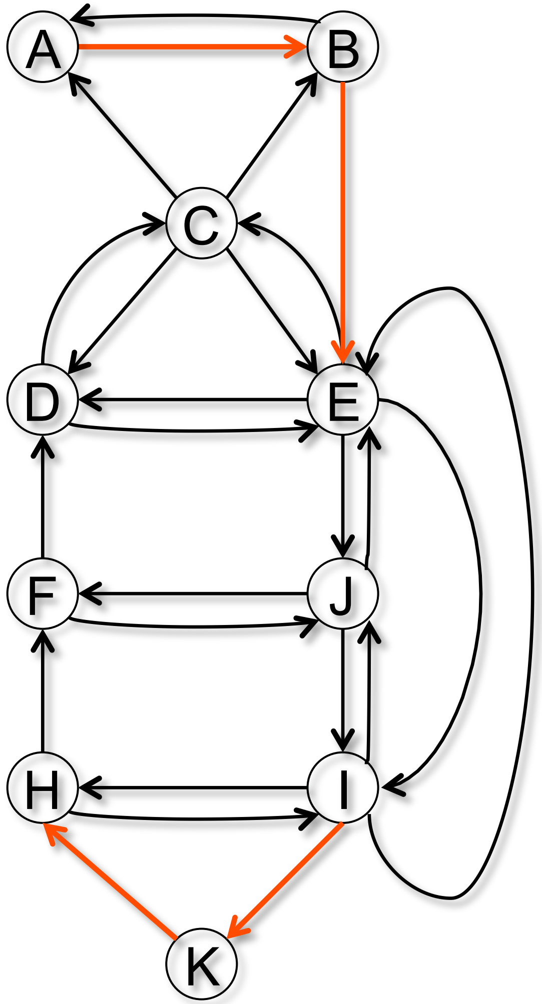

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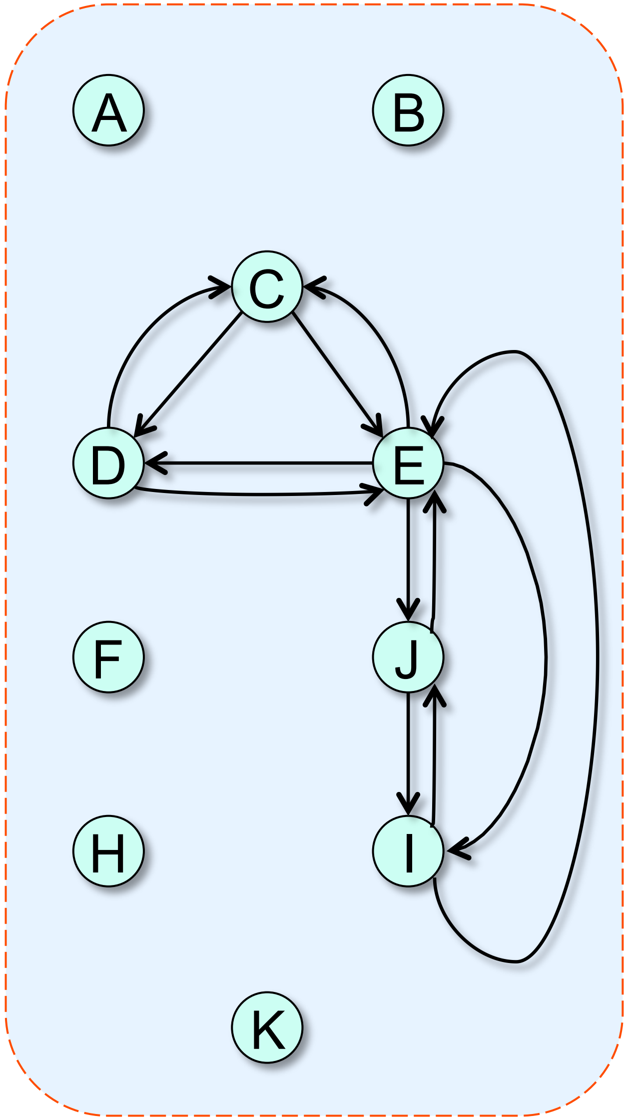

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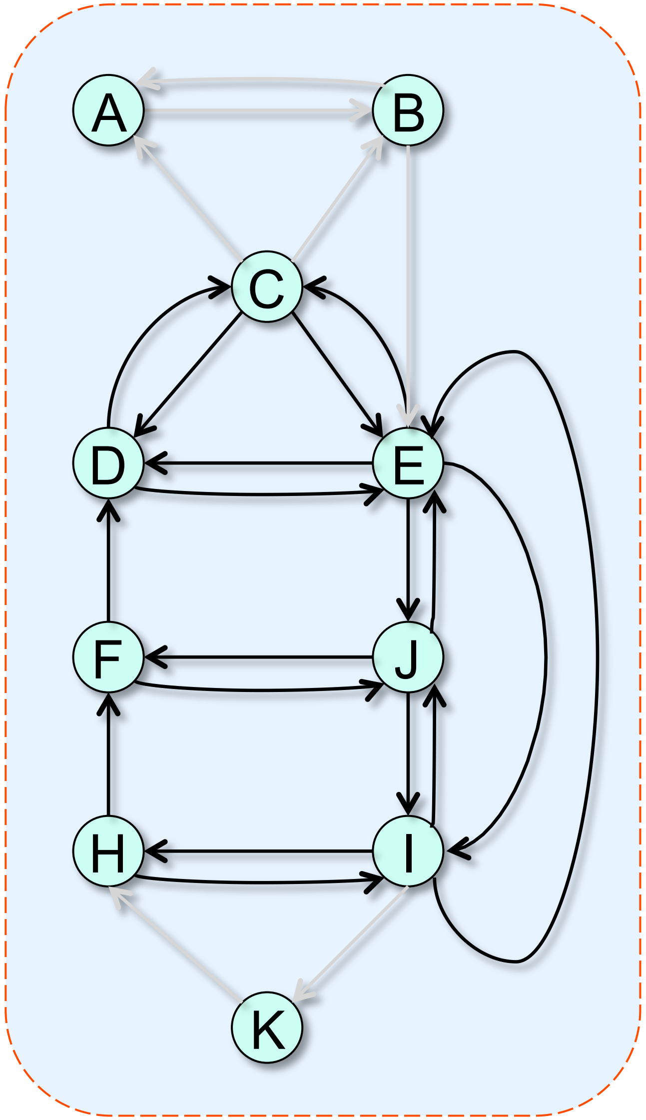
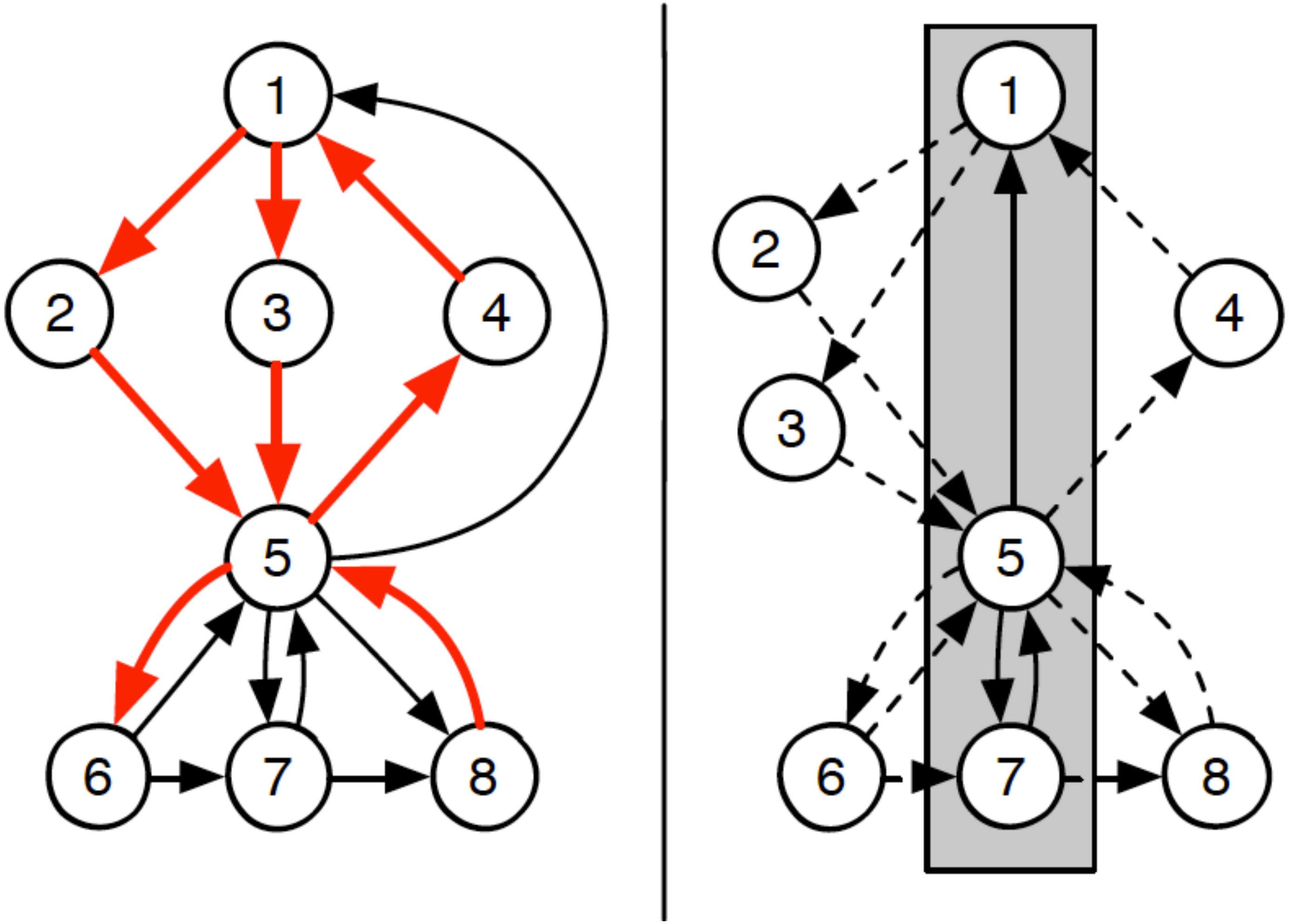


2-Edge-Connected Blocks

How easy is it to compute 2-edge-connected blocks?

Can we just remove strong bridges?



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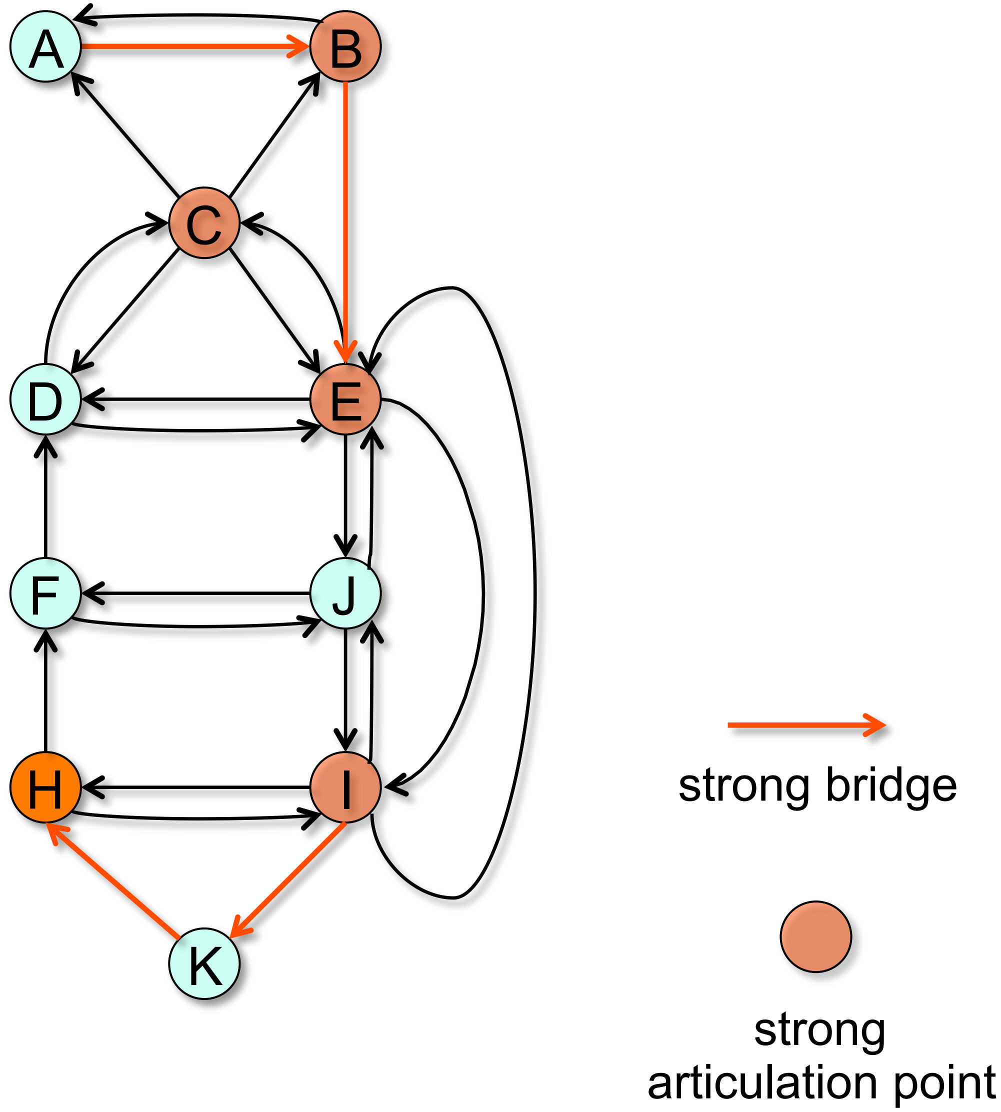

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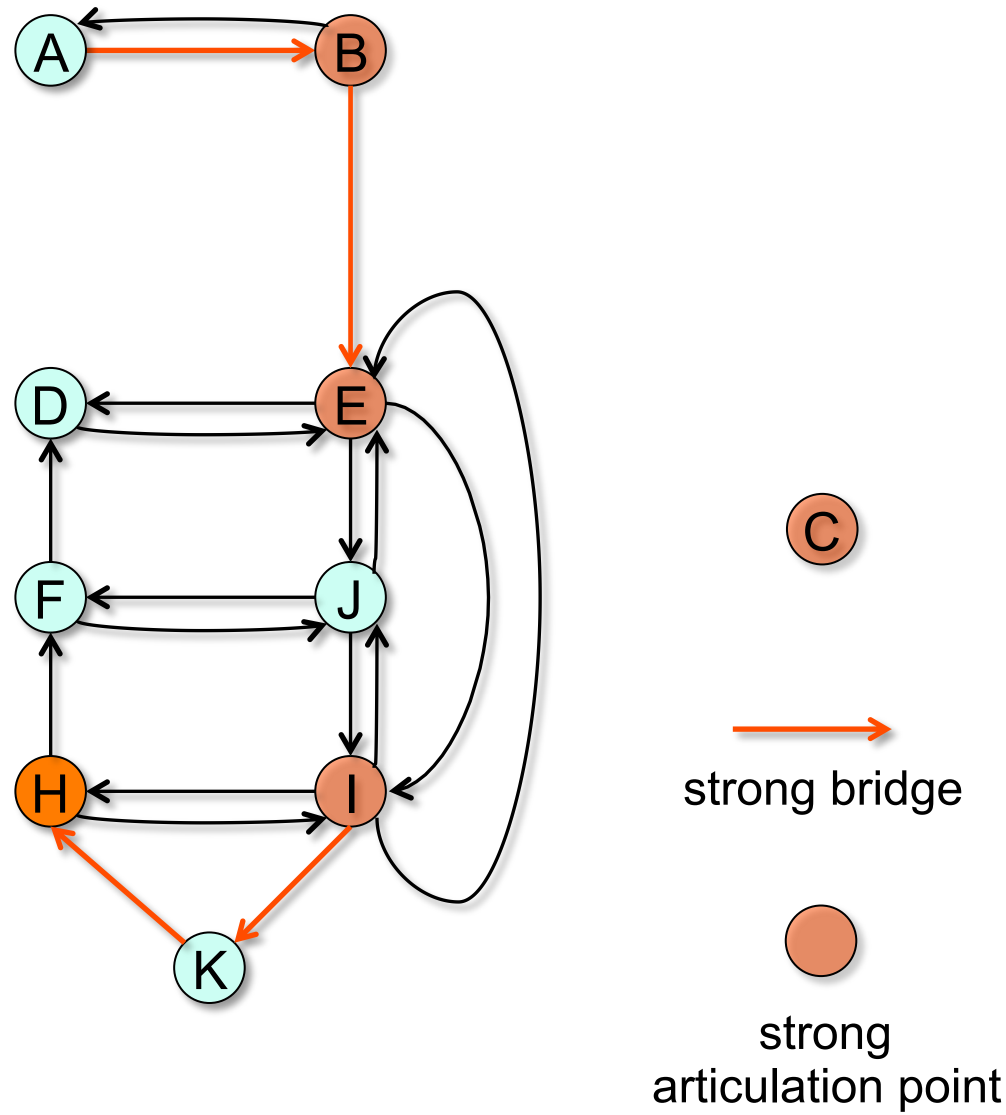

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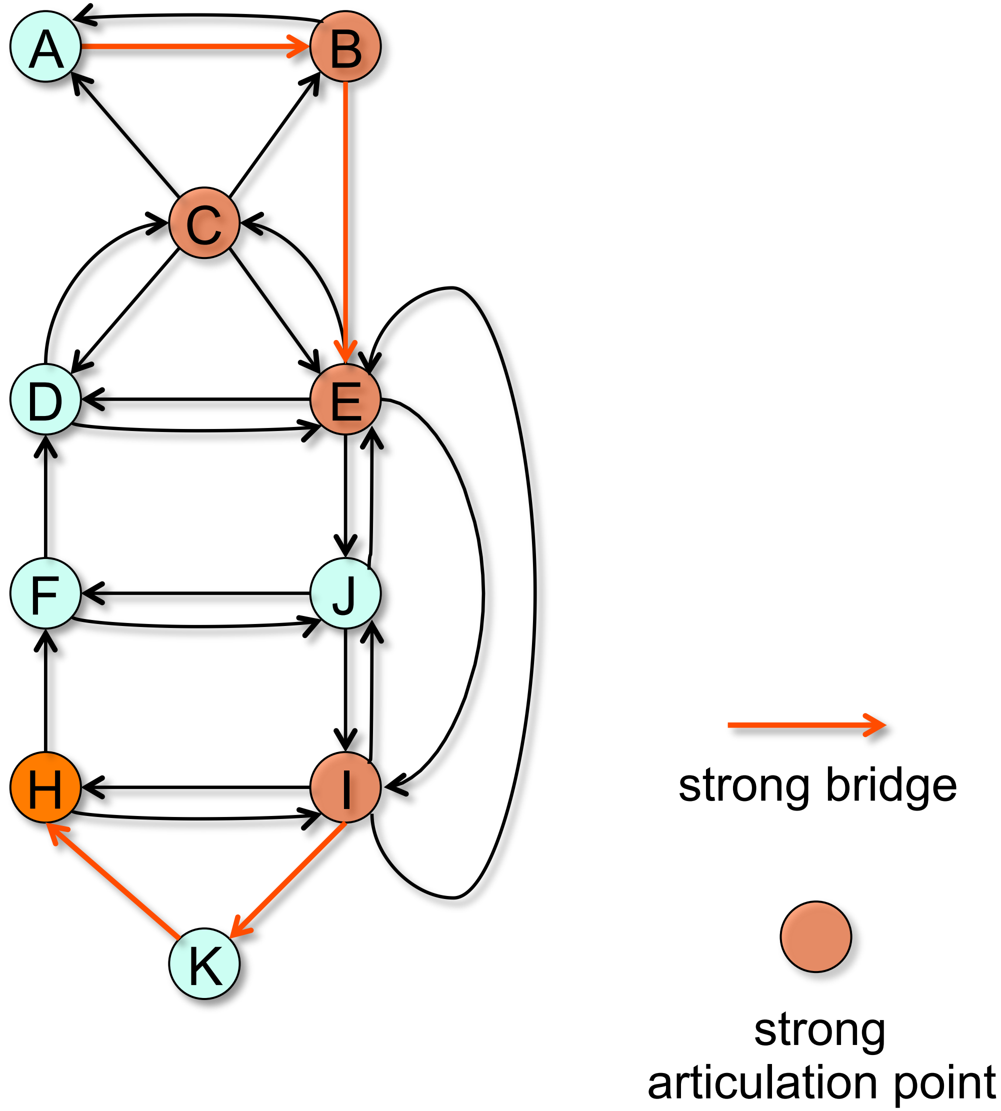

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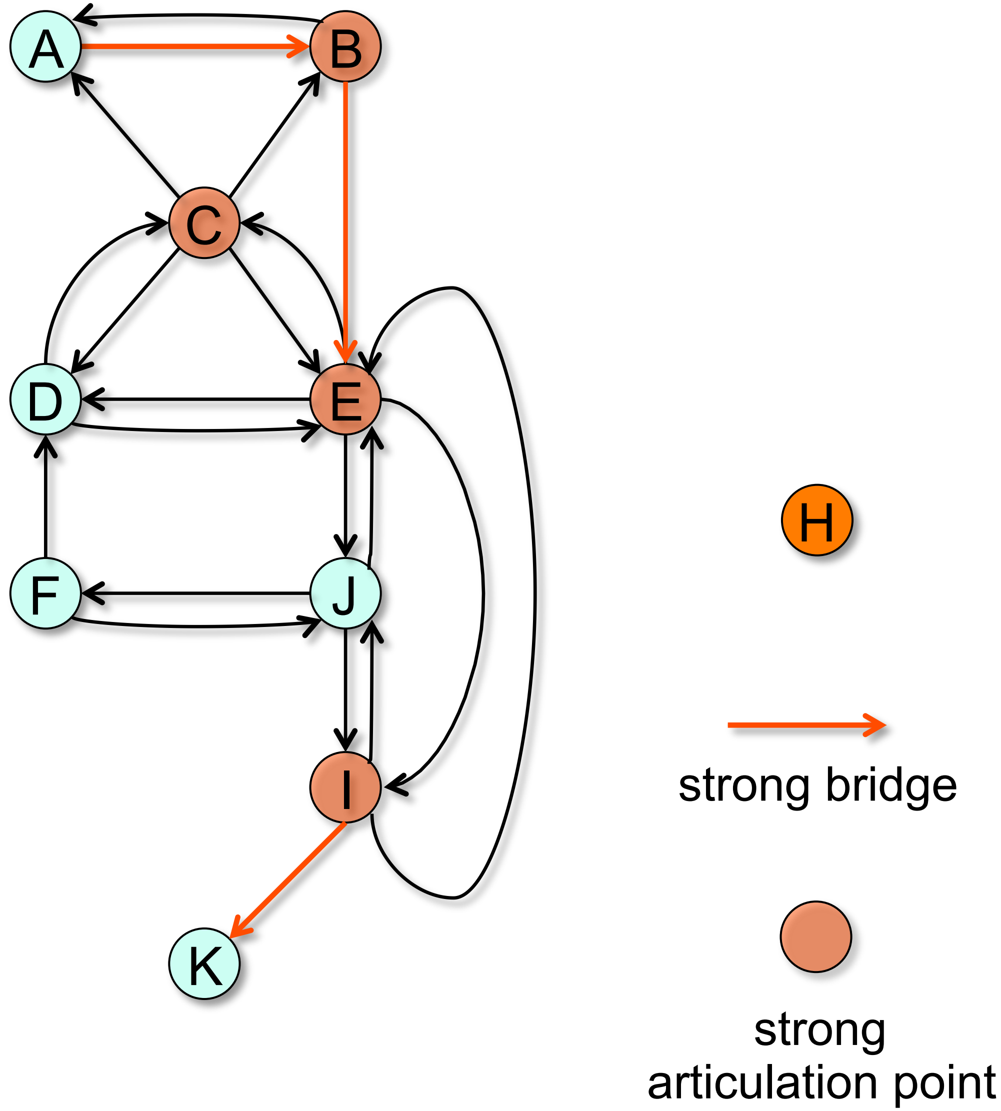

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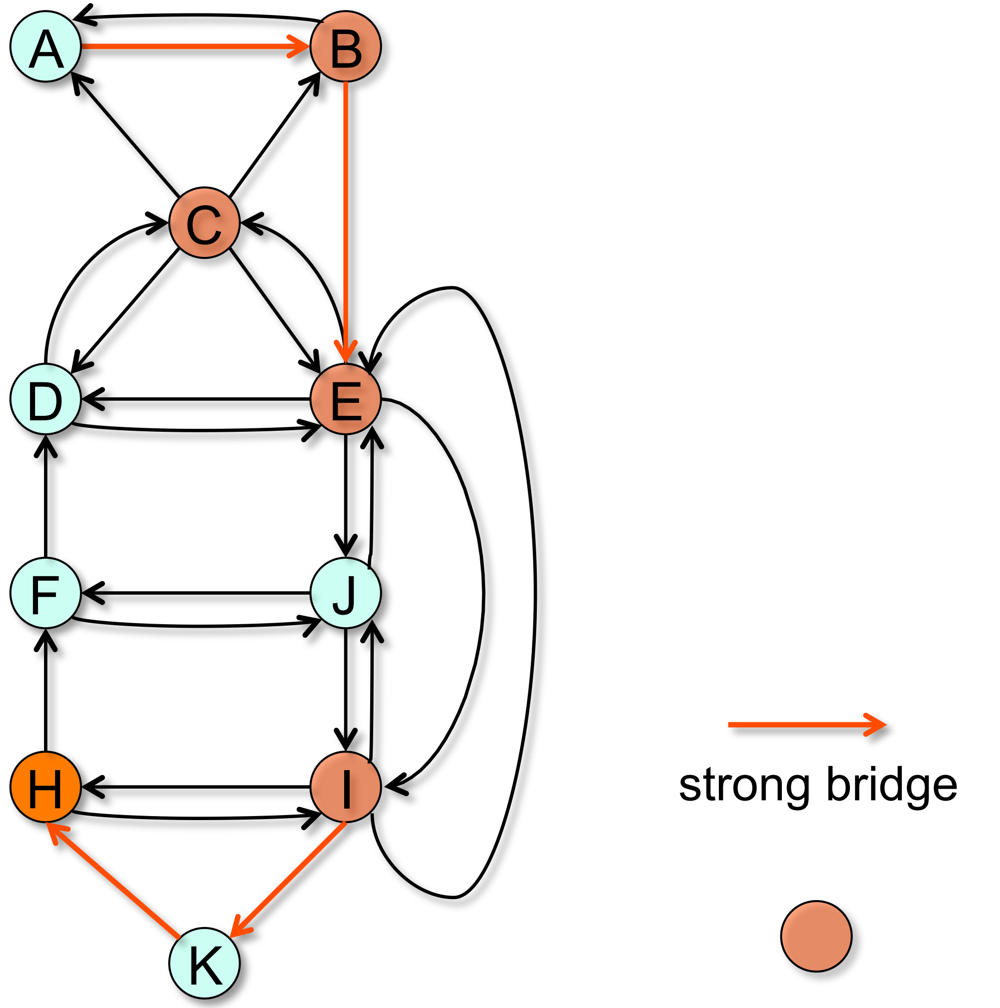

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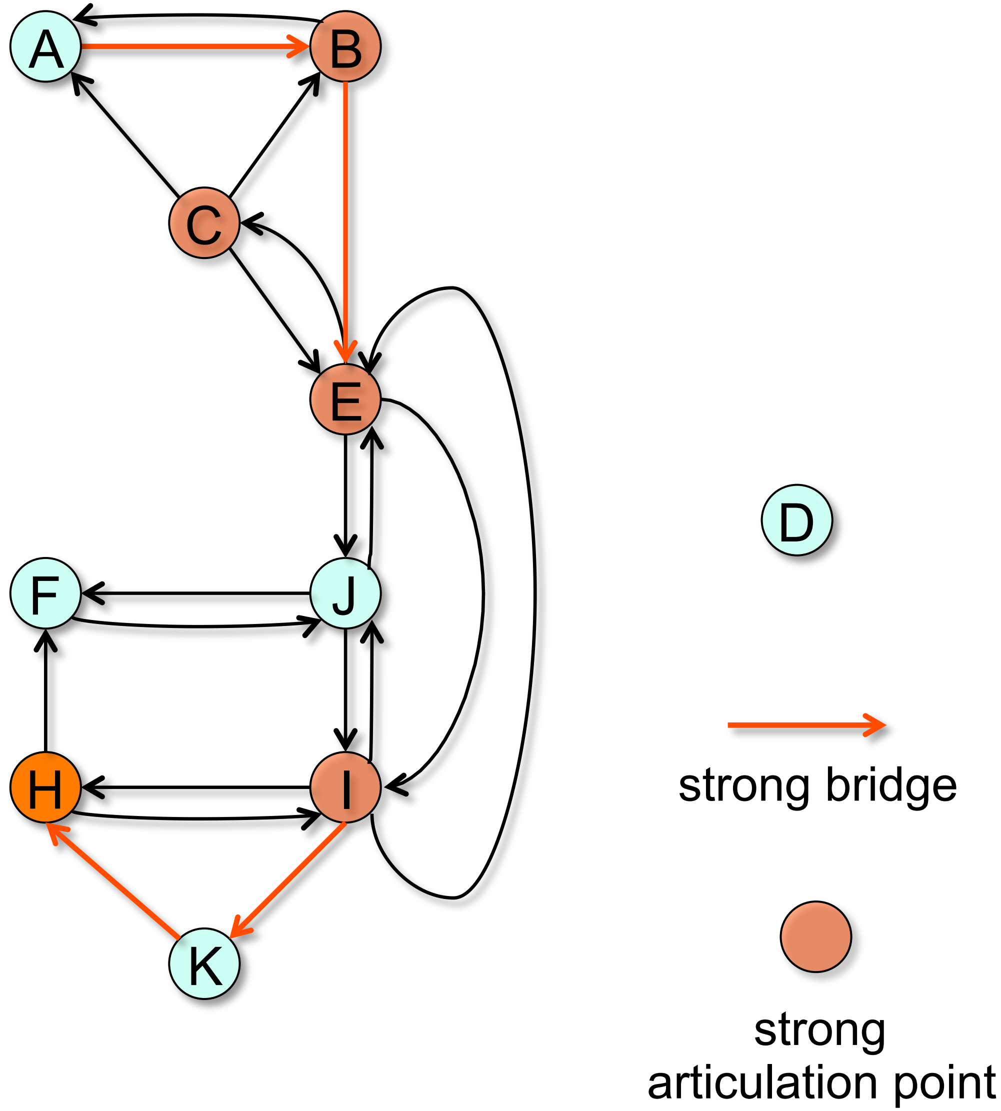

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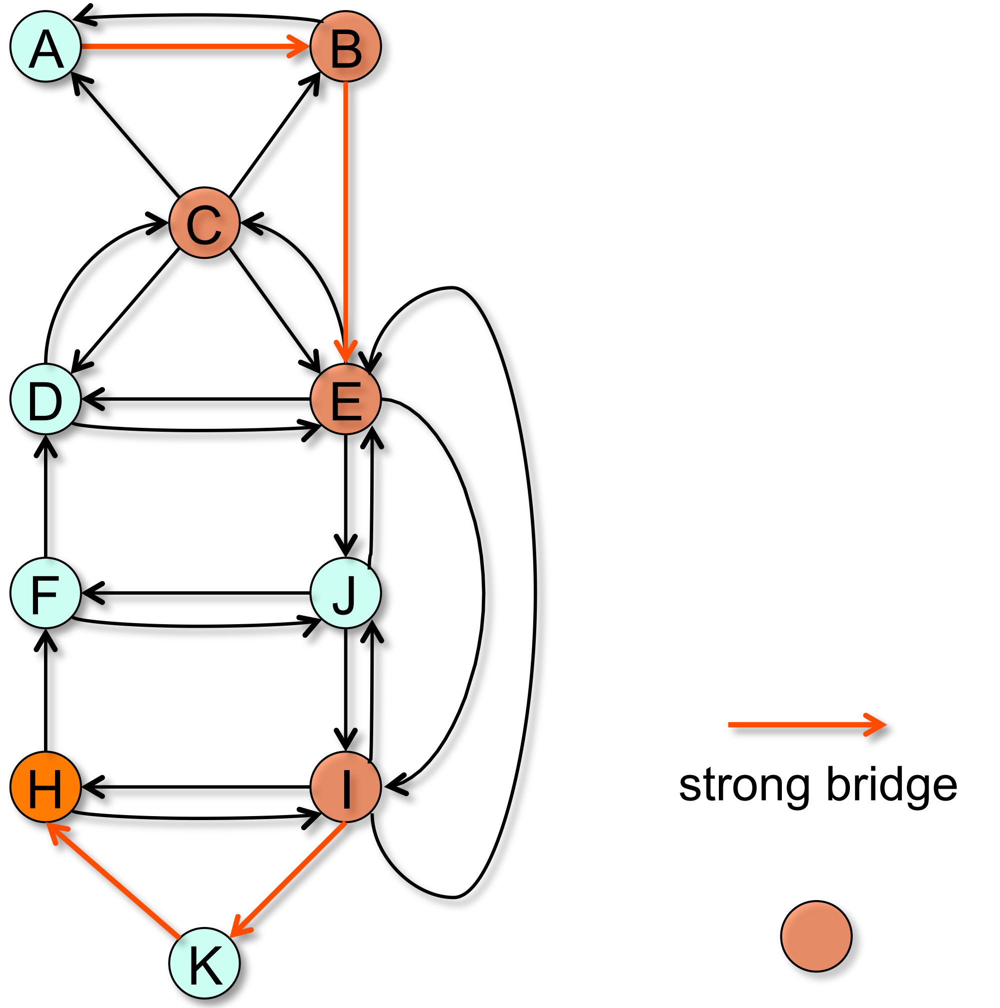

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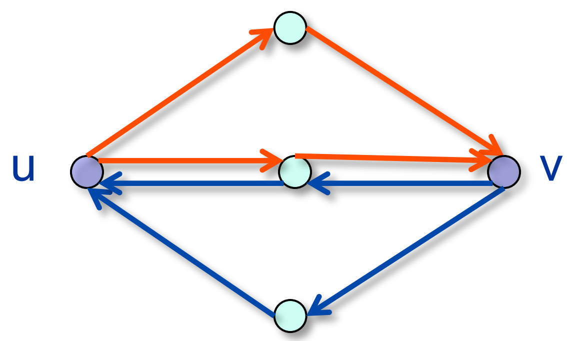

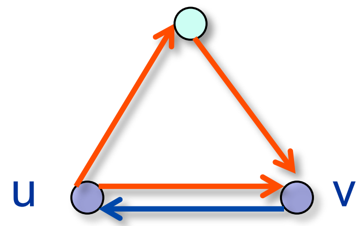

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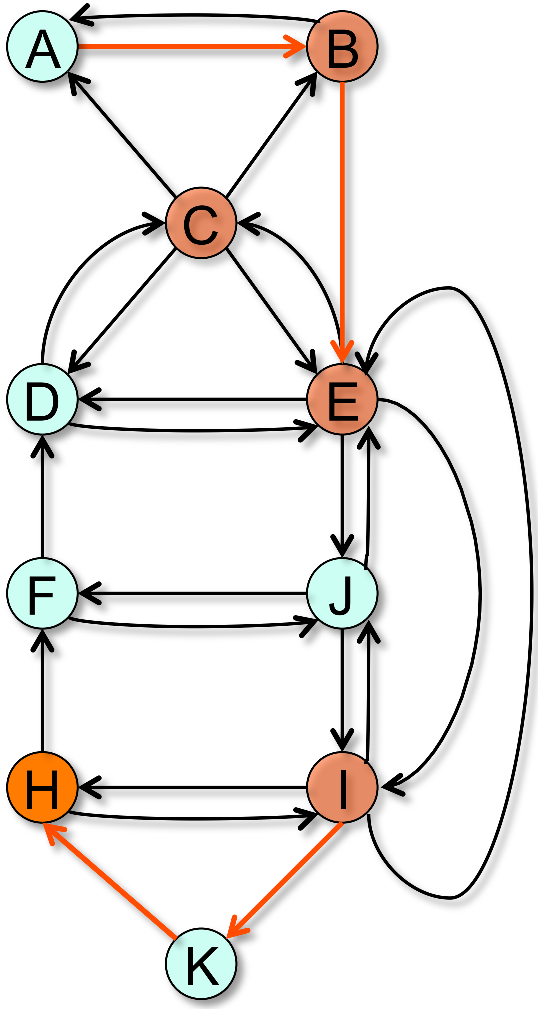
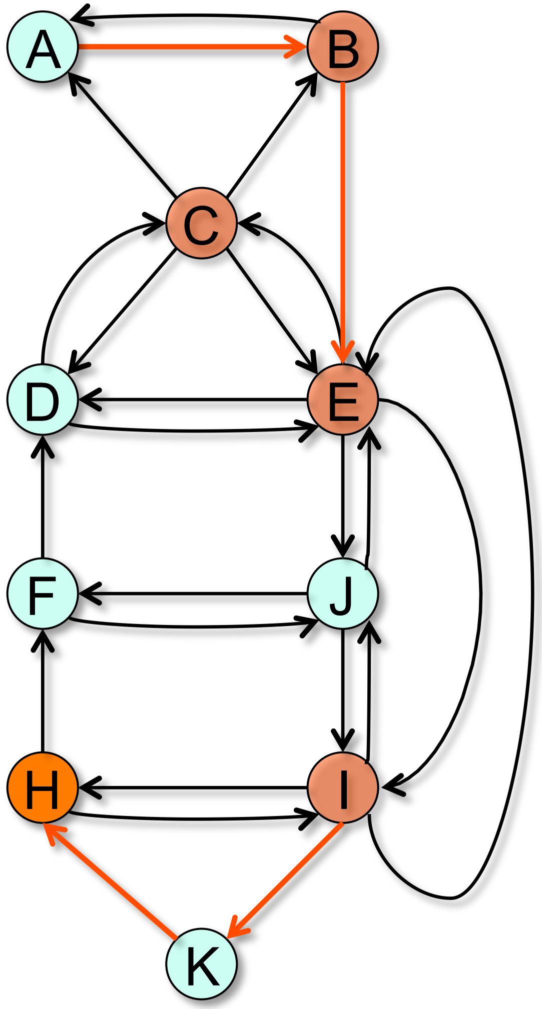
2-Vertex-Connected Comp's ≠ Blocks



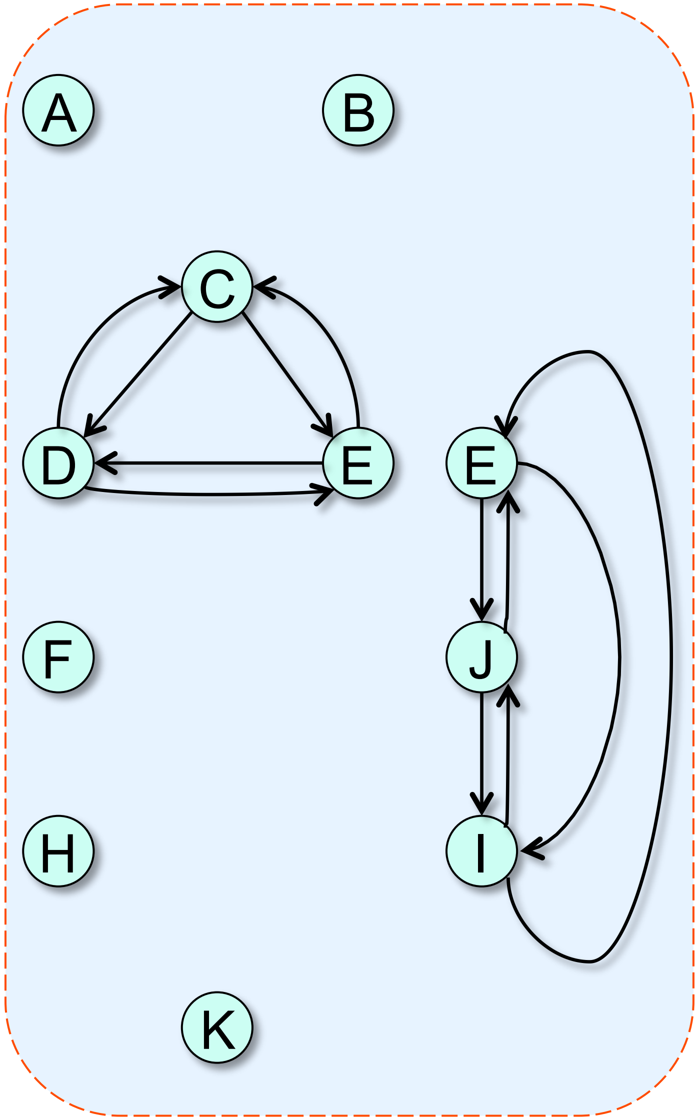
strong bridge

strong
articulation point

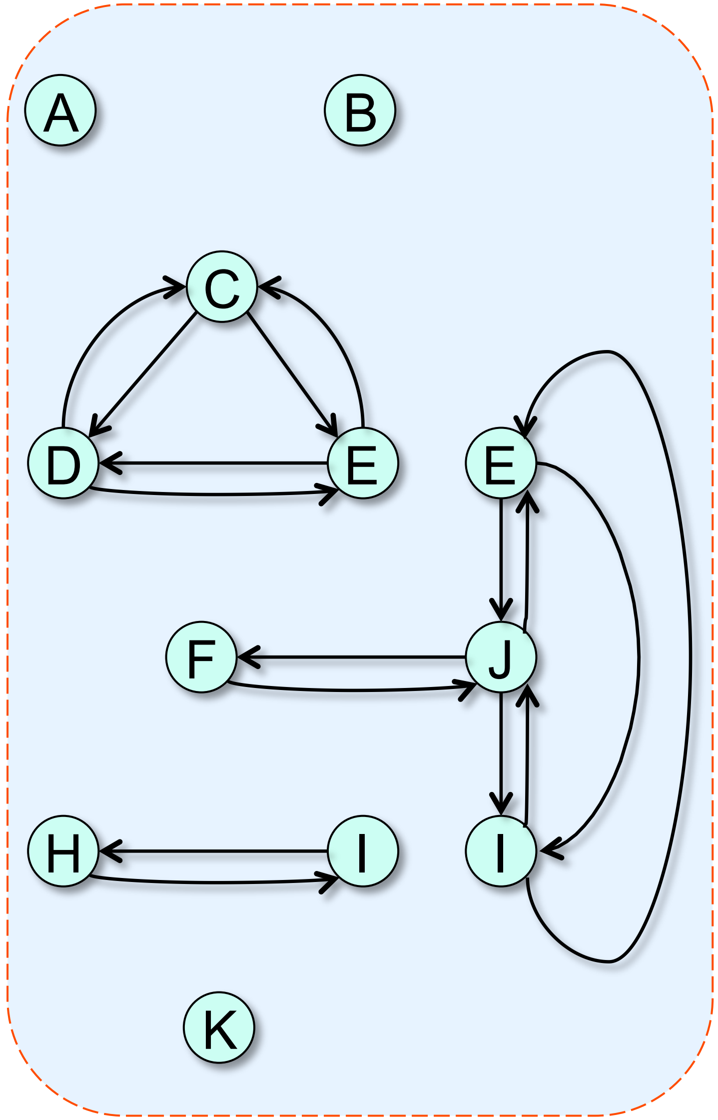
2-Vertex-Connected Comp's ≠ Blocks



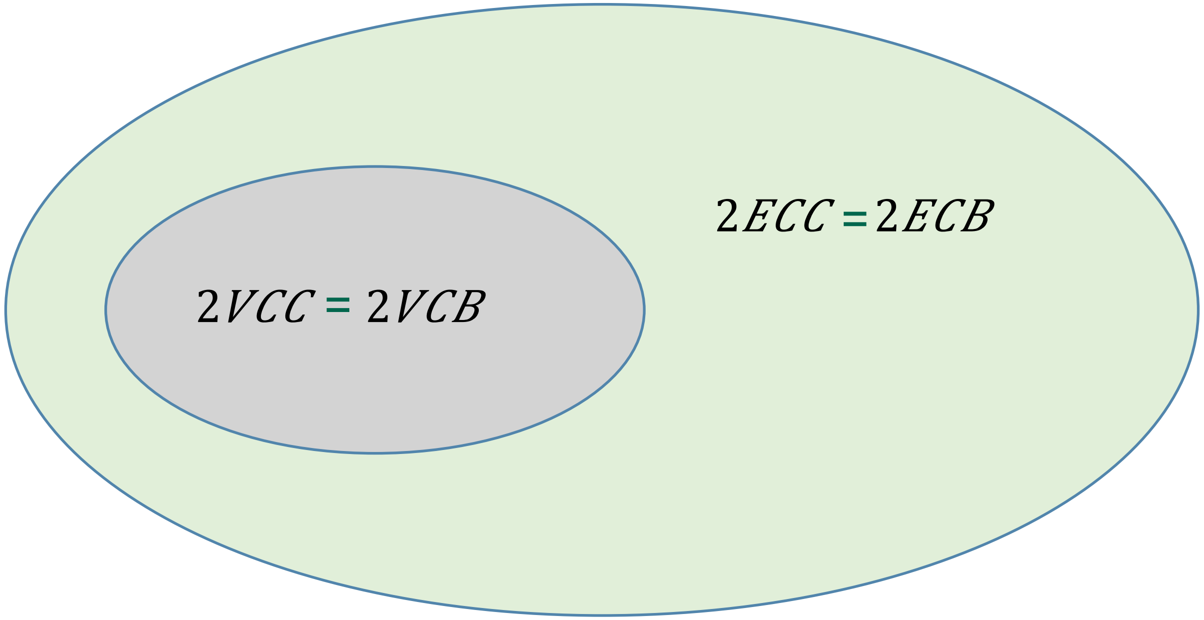
2-vertex-connected components



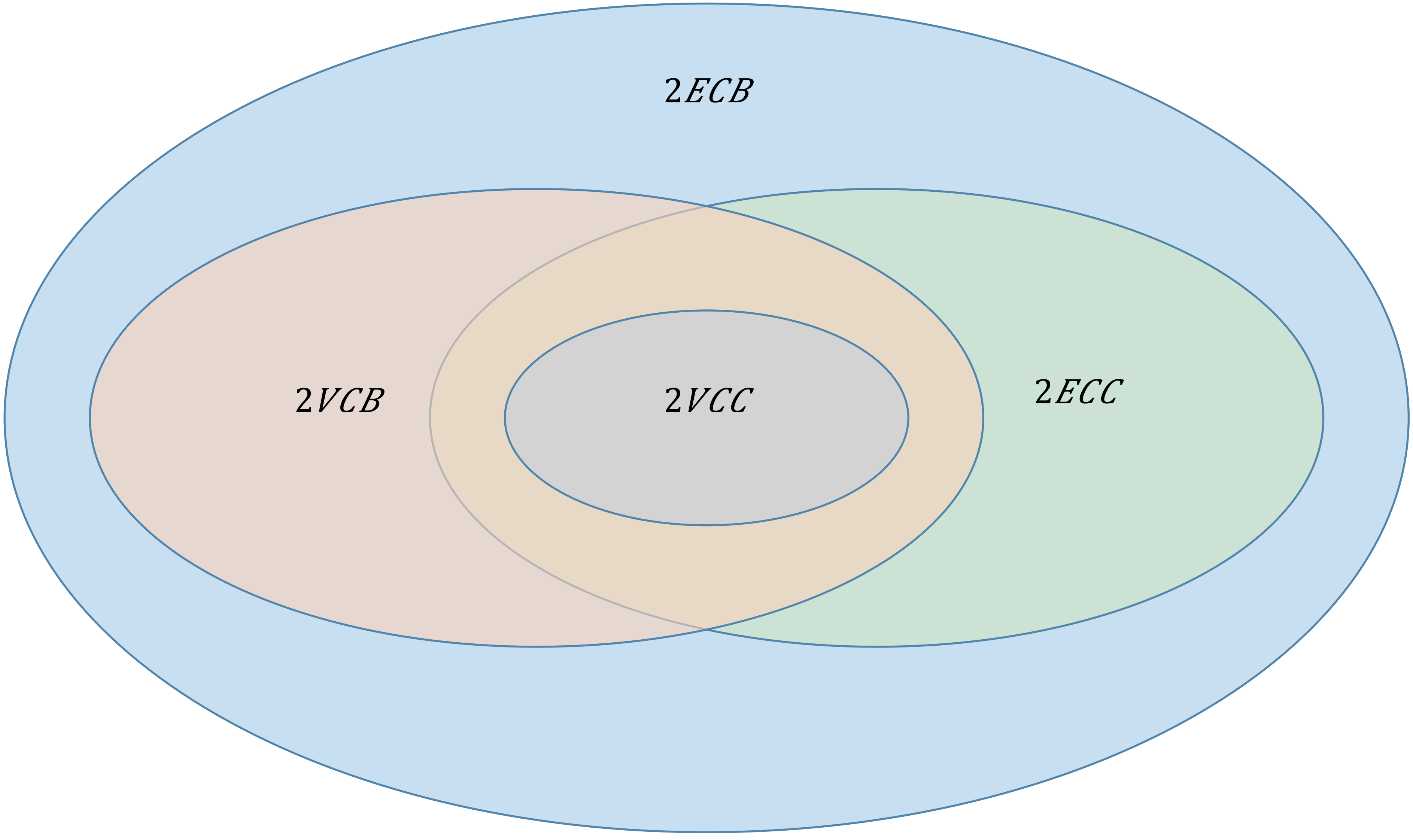
2-vertex-connected blocks



Big Picture (Undirected)



Big Picture (Directed)



Bounds for Directed *G*

Q1: Find whether *G* is 2-vertex-connected (2-edge-connected). I.e., find **one** connectivity cut (if any)

O(m+n)

[Tarjan 76] + [Gabow & Tarjan 83]

[Georgiadis 10]

Q2: Find **all** 2-connectivity cuts (articulation points, bridges) in *G*

O(m+n)

[Italiano et al 10]

Q3: Find the **2-connectivity** (2-vertex-, 2-edge- connected) **blocks** of *G*

O(m+n)

[Georgiadis et al 15]

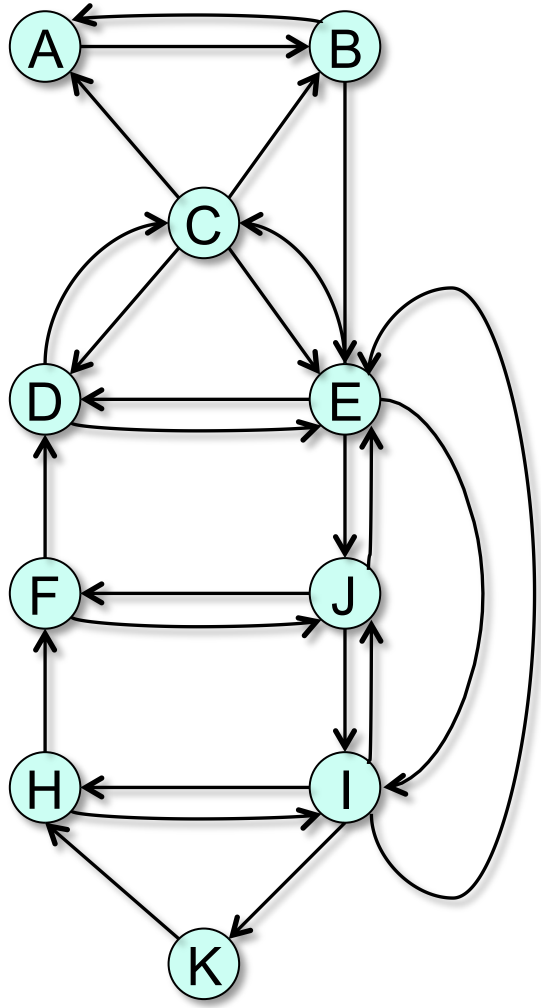
Q4: Find the **2-connectivity** (2-vertex-, 2-edge-connected) **components** of *G*

O(mn)

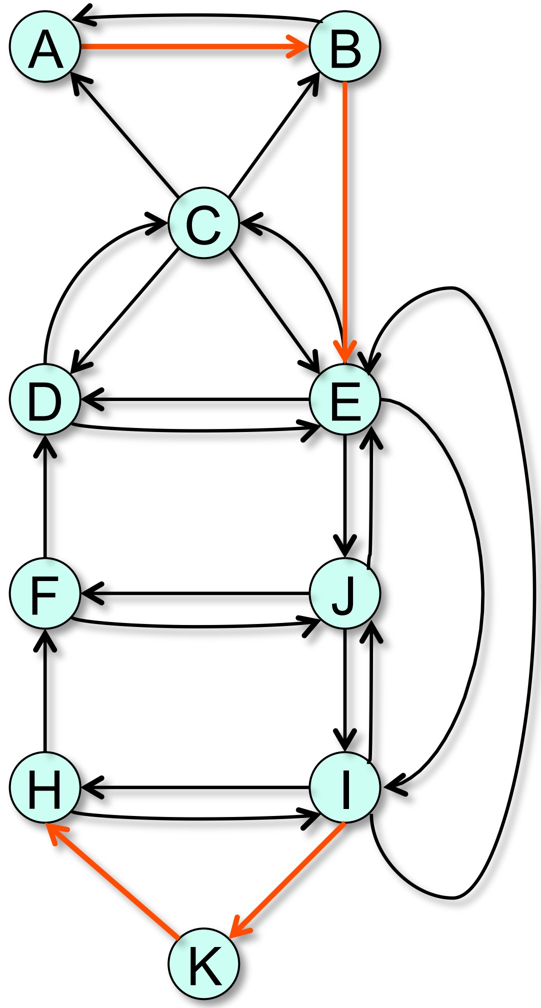
[Jaberi 14]

Can we do better?

2-Edge Connected Components



2-Edge Connected Components



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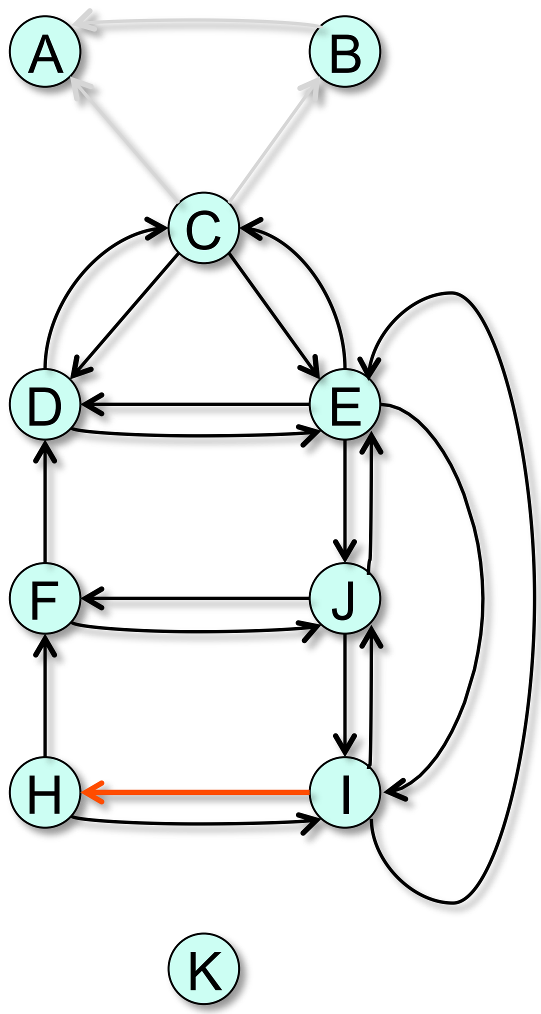

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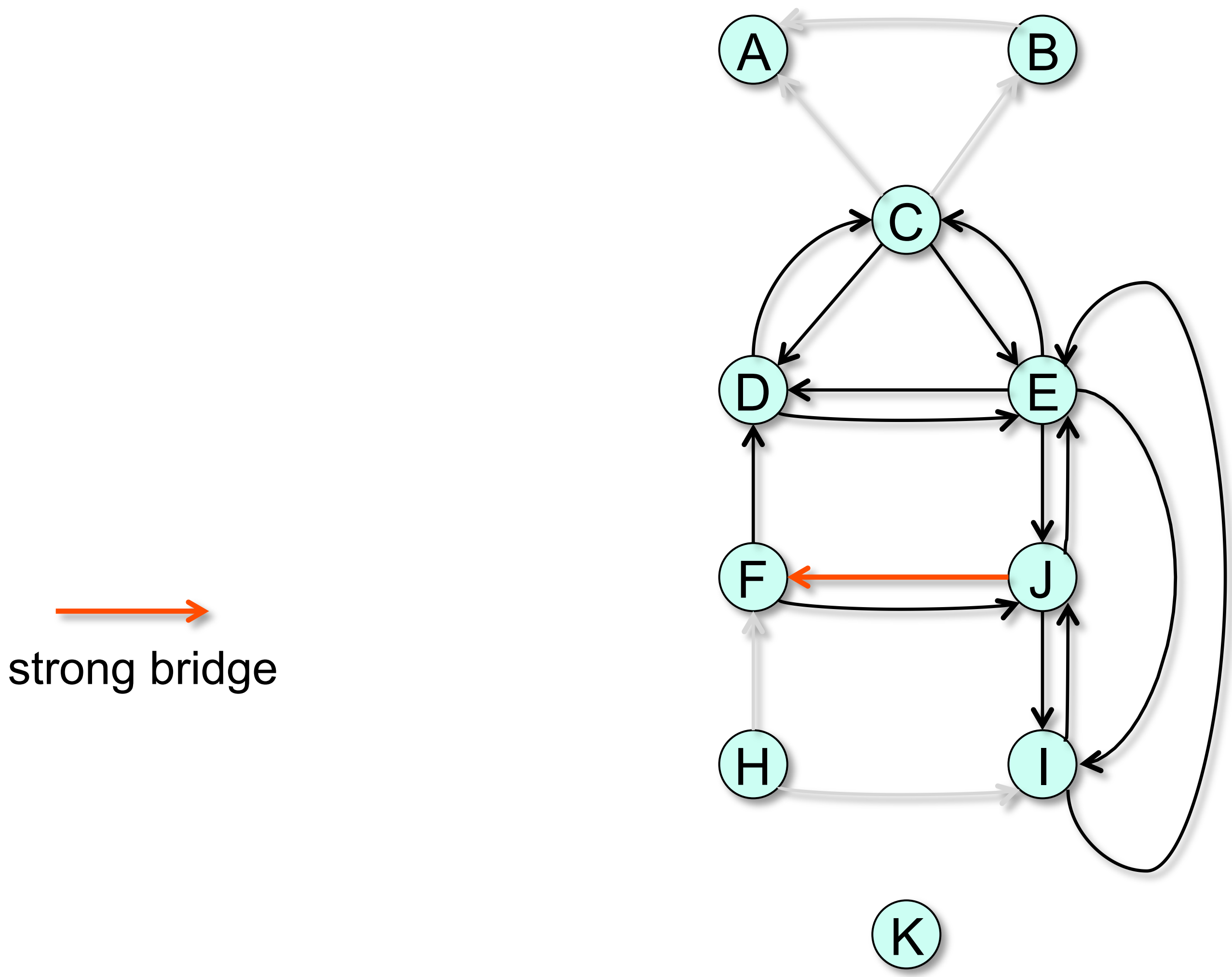
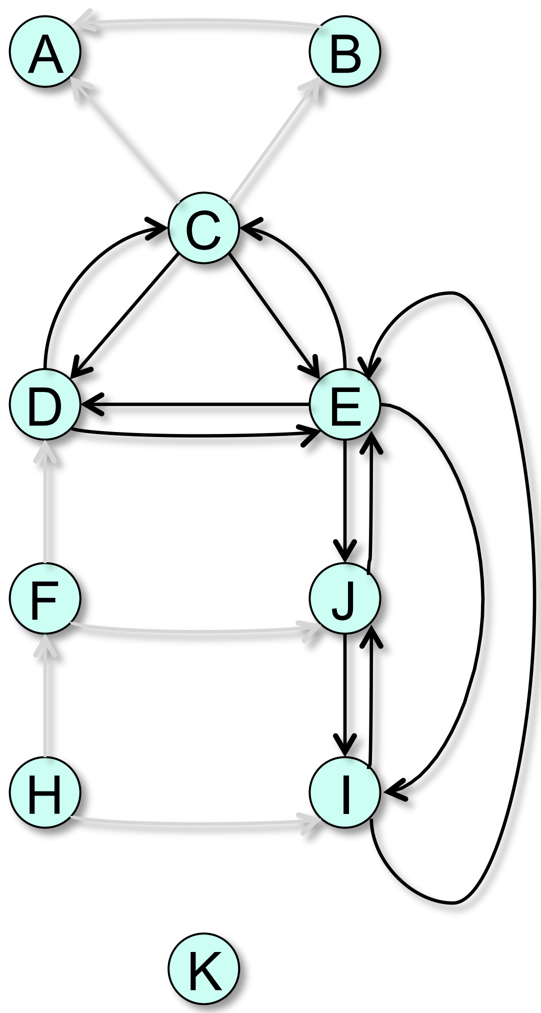
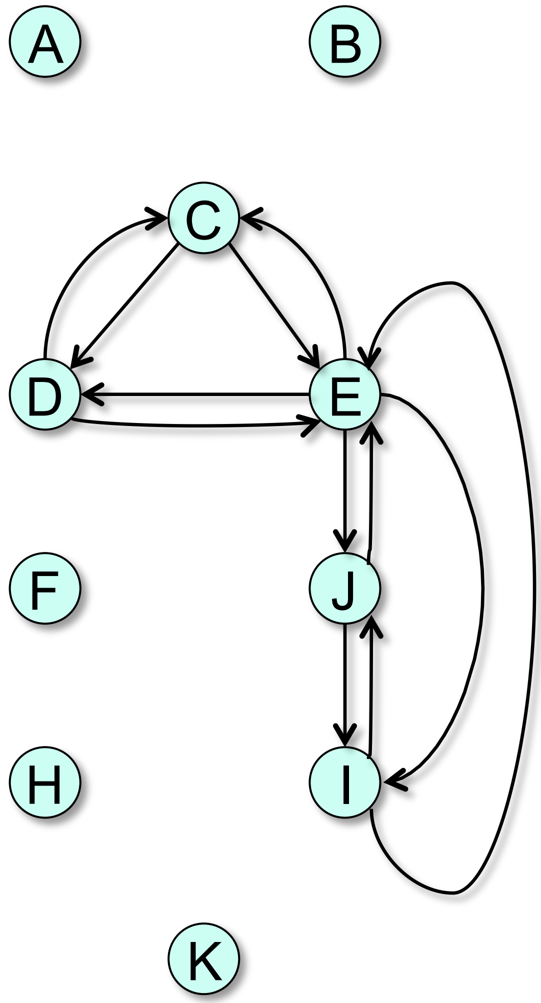


2-Edge Connected Components



2-Edge Connected Components



Why should we care?

Theoretically interesting problem

It also has several intriguing applications

Bounds for Directed *G*

Q1: Find whether *G* is 2-vertex-connected (2-edge-connected).
I.e., find **one** connectivity cut (if any)

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Q4: Find the **2-connectivity** (2-vertex-, 2-edge-connected) **components** of *G*

O(mn)

[Jaberi 14]

Today's Outline

1. 2-Connectivity on directed graphs
2. **Algorithms for strong articulation points and strong bridges**
3. Experiments
4. Open Problems

Naive Algorithms

Check whether vertex v is strong articulation point in G :

Compute strongly connected components of $G/\{v\}$

$O(n(m+n))$ for computing all strong articulation points

Check whether edge e is strong bridge in G :

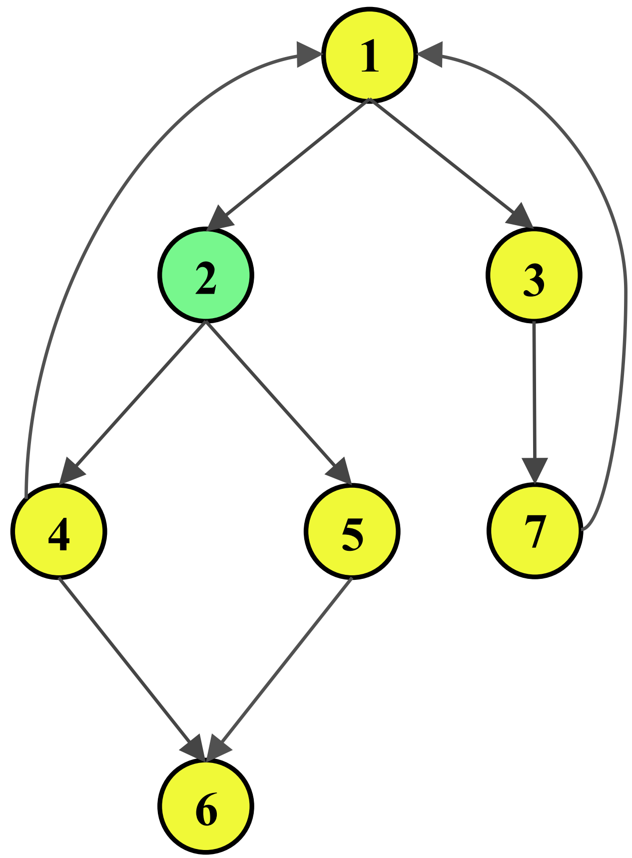
Compute strongly connected components of $G/\{e\}$

$O(m(m+n))$ for computing all strong bridges

Not difficult to get $O(n(m+n))$ algorithm

Flow graphs and Dominators

A *flow graph $G(s) = (V,E,s)$* is a directed graph with a start vertex s in V such that every vertex in V reachable from s



Flow graphs and Dominators

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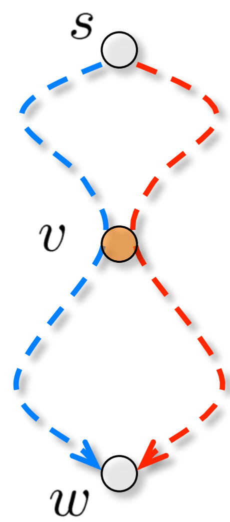

Given a flow graph *G(s)=(V,E,s)*, can define a *dominance relation*: vertex *v* *dominates* vertex *w* if every path from *s* to *w* includes *v*

Flow graphs and Dominators

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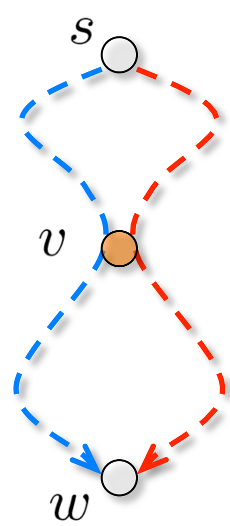


Given a flow graph *G(s)=(V,E,s)*, can define a *dominance relation*: vertex *v* *dominates* vertex *w* if every path from *s* to *w* includes *v*

Let *dom(w)* be set of vertices that dominate *w*. For any $w \neq s$ we have that $\{s,w\} \subseteq dom(w)$: *s* and *w* are the *trivial dominators* of *w*

Dominator Trees

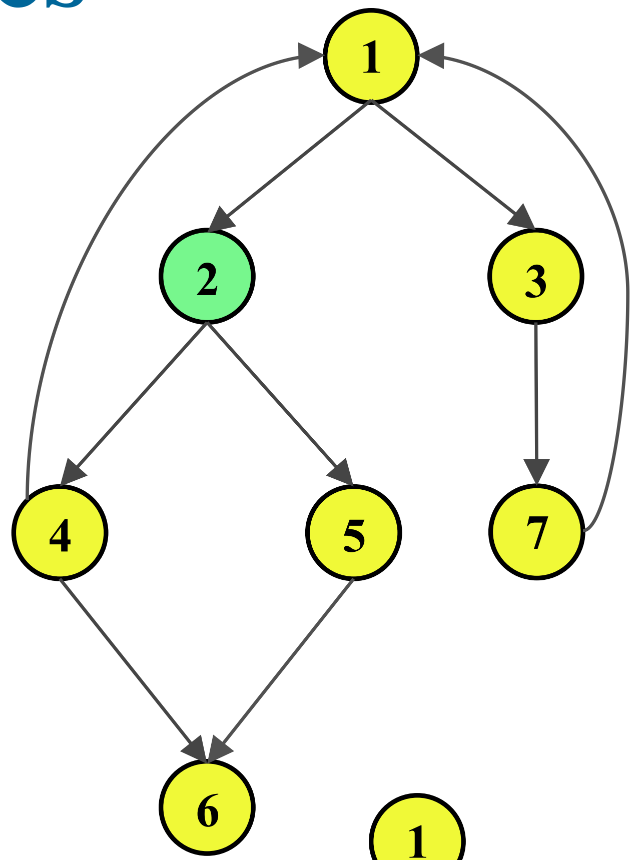
Dominance relation is transitive and its transitive reduction is referred to as the *dominator tree DT(s)*.

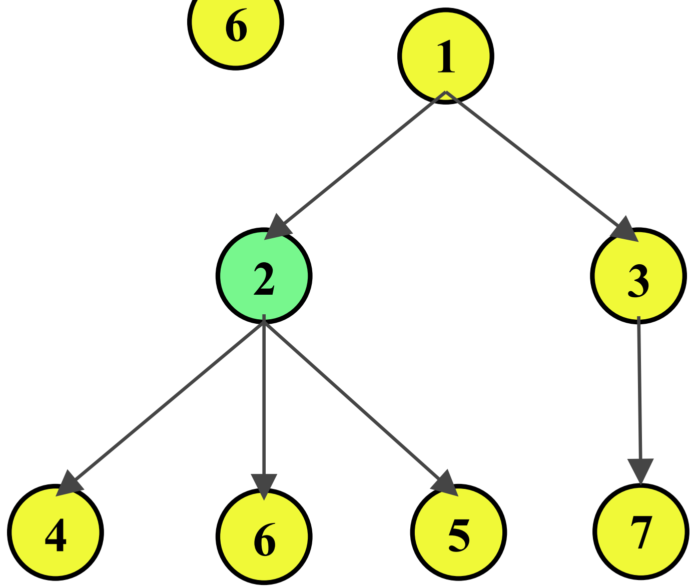
DT(s) rooted at *s*.

v dominates *w* if and only if *v* is ancestor of *w* in dominator tree *DT(s)*.

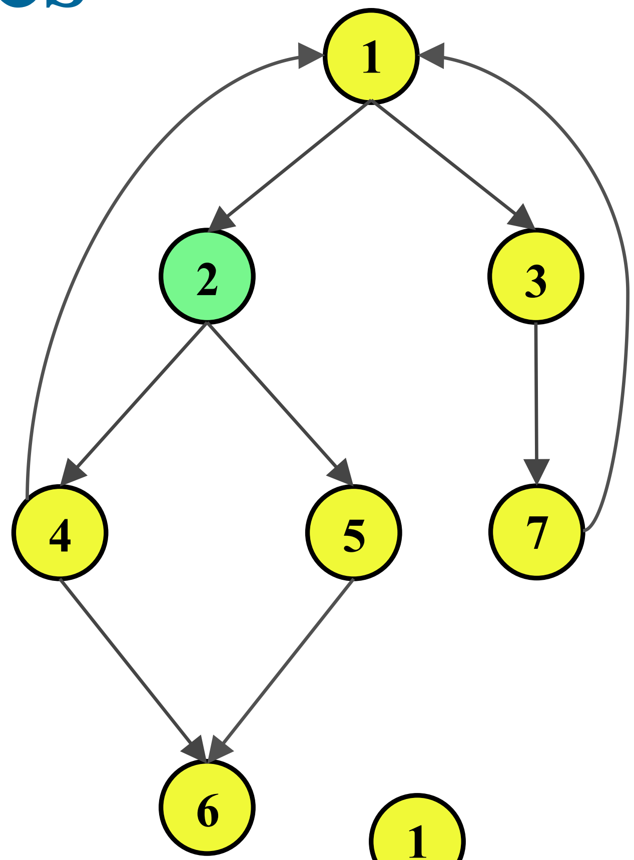
If *v* dominates *w*, and every other non-trivial dominator of *v* also dominates *w*, *v* is an *immediate dominator* of *w*.

If *v* has any non-trivial dominators, then *v* has a unique immediate dominator: the immediate dominator of *v* is the parent of *v* in the dominator tree *DT(s)*.





Dominator Trees

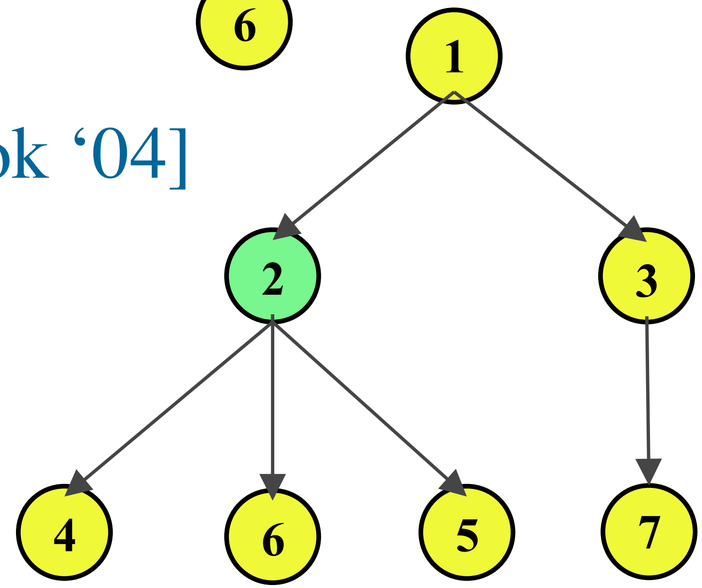


Dominators (and dominator tree) can be computed in $O(m+n)$ time:

[Alstrup, Harel, Lauridsen &Thorup '97]

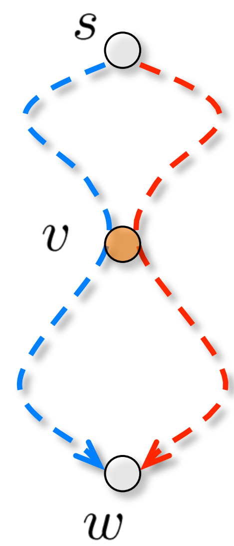
[Buchsbaum, Kaplan, Rogers & Westbrook '04]

[Georgiadis & Tarjan '04]



Vertex Dominators and SAP

Lemma 1 *Let $G = (V,E)$ be a strongly connected graph, and let s be any vertex in G. Let $G(s) = (V,E,s)$ be the flow graph with start vertex s. If v is a non-trivial dominator of a vertex w in $G(s)$, then v is a strong articulation point in G.*



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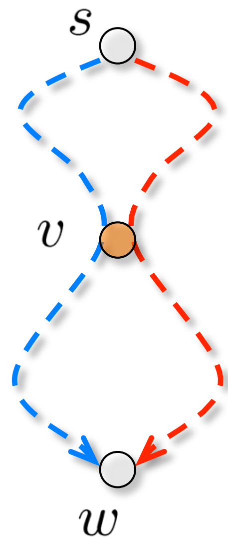
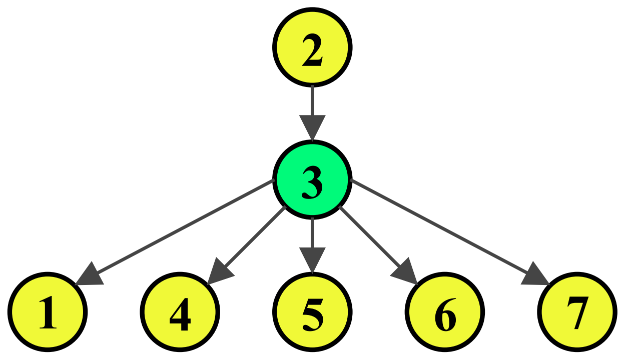
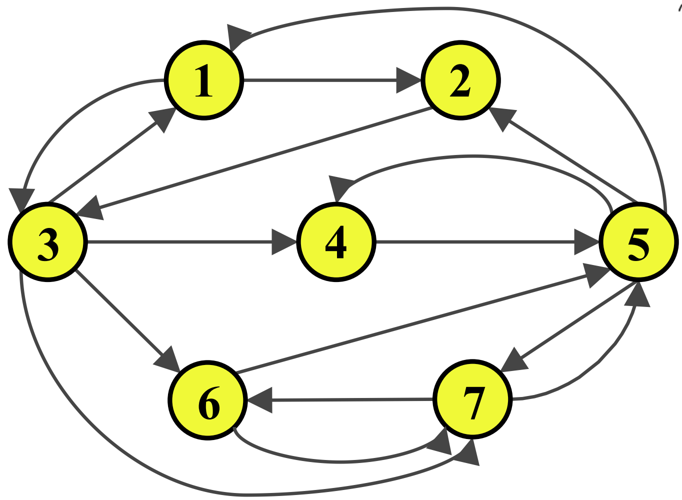




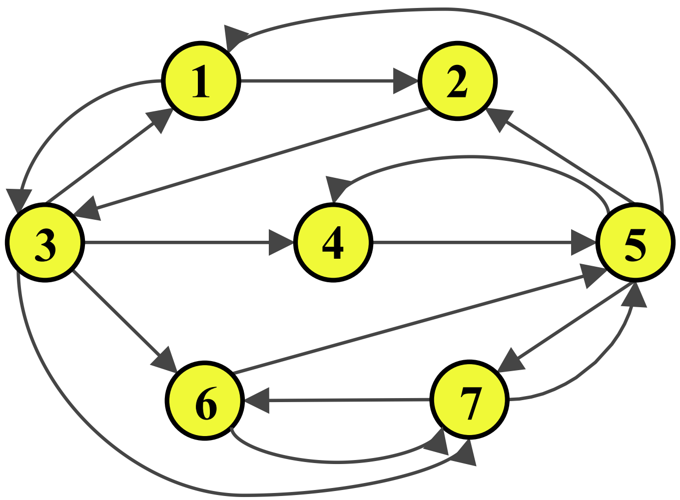
Vertex 3 is a non-trivial dominator in G(2)

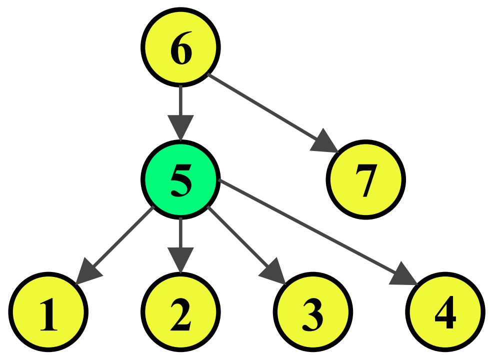


Vertex 3 is strong articulation point in G

Vertex Dominators and SAP

Lemma 2 *Let G = (V,E) be a strongly connected graph. If v is a strong articulation point in G, then there must be a vertex s ∈ V such that v is a non-trivial dominator of a vertex w in the flow graph G(s) = (V,E,s).*





Vertex 5 is strong articulation point in G

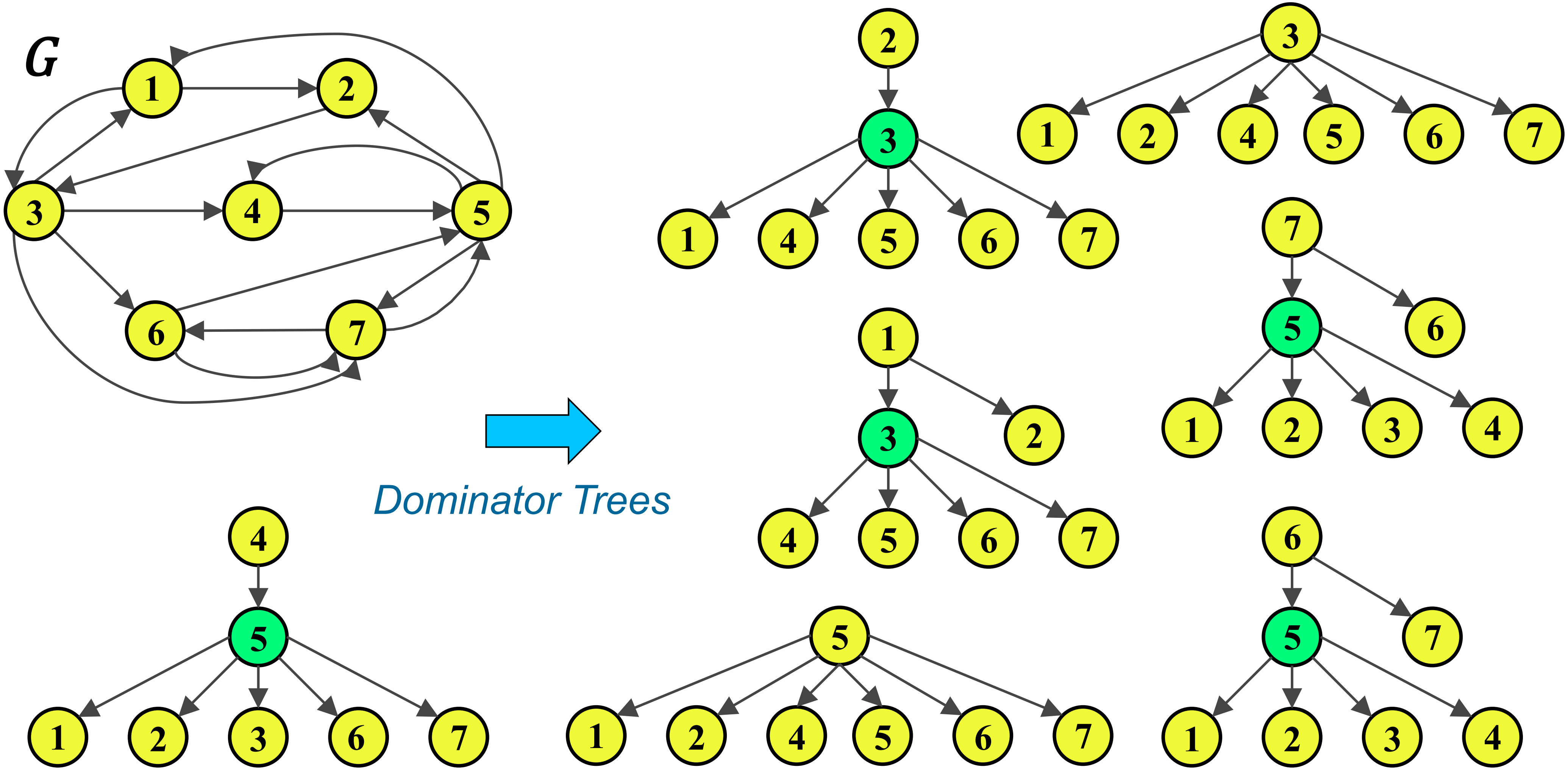
Vertex 5 must be non-trivial dominator in some G(s). Here s=6.

Still Not Efficient

Corollary *Let $G = (V,E)$ be a strongly connected graph. Vertex u is a strong articulation point in G if and only there is a vertex $s \in V$ such that u is a non-trivial dominator of a vertex v in the flow graph $G(s) = (V,E,s)$.*

Must compute dominator trees for all flow graphs G(v), for each vertex v in V, and output all non-trivial dominators found.

Dominator Trees



Still Not Efficient

Corollary *Let $G = (V,E)$ be a strongly connected graph. Vertex u is a strong articulation point in G if and only then there is a vertex $s \in V$ such that u is a non-trivial dominator of a vertex v in the flow graph $G(s) = (V,E,s)$.*

Must compute dominator trees for all flow graphs G(v), for each vertex v in V, and output all non-trivial dominators found.

Takes $O(n(m+n))$ time

Like trivial algorithm

Only more complicated...

Reversal Graph

Reversal Graph $G^R = (V,E^R)$: reverse all edges in G.
If (u,v) in G then (v,u) in G^R.

Observation. *Let $G = (V,E)$ be a strongly connected graph and $G^R = (V,E^R)$ be its reversal graph. Then G^R is strongly connected. Furthermore, vertex v is a strong articulation point in G if and only if v is a strong articulation point in G^R.*

Exploit Dominators

Given a strongly connected graph $G=(V,E)$, let

- $G(s) = (V,E,s)$ be the flow graph with start vertex s
- $D(s)$ the set of non-trivial dominators in $G(s)$
- $G^R(s) = (V,E^R,s)$ be the flow graph with start vertex s
- $D^R(s)$ the set of non-trivial dominators in $G^R(s)$

Theorem. *Let $G = (V,E)$ be a strongly connected graph, and let $s \in V$ be any vertex in G. Then vertex $v \neq s$ is a strong articulation point in G if and only if $v \in D(s) \cup D^R(s)$.*

Strong Articulation Points

Theorem. *Let $G = (V,E)$ be a strongly connected graph, and let $s \in V$ be any vertex in G. Then vertex $v \neq s$ is a strong articulation point in G if and only if $v \in D(s) \cup D^R(s)$.*

Proof:

If $v \in D(s) \cup D^R(s)$ we know from previous lemmas that v must be an articulation point.

So, we need to prove only one direction.

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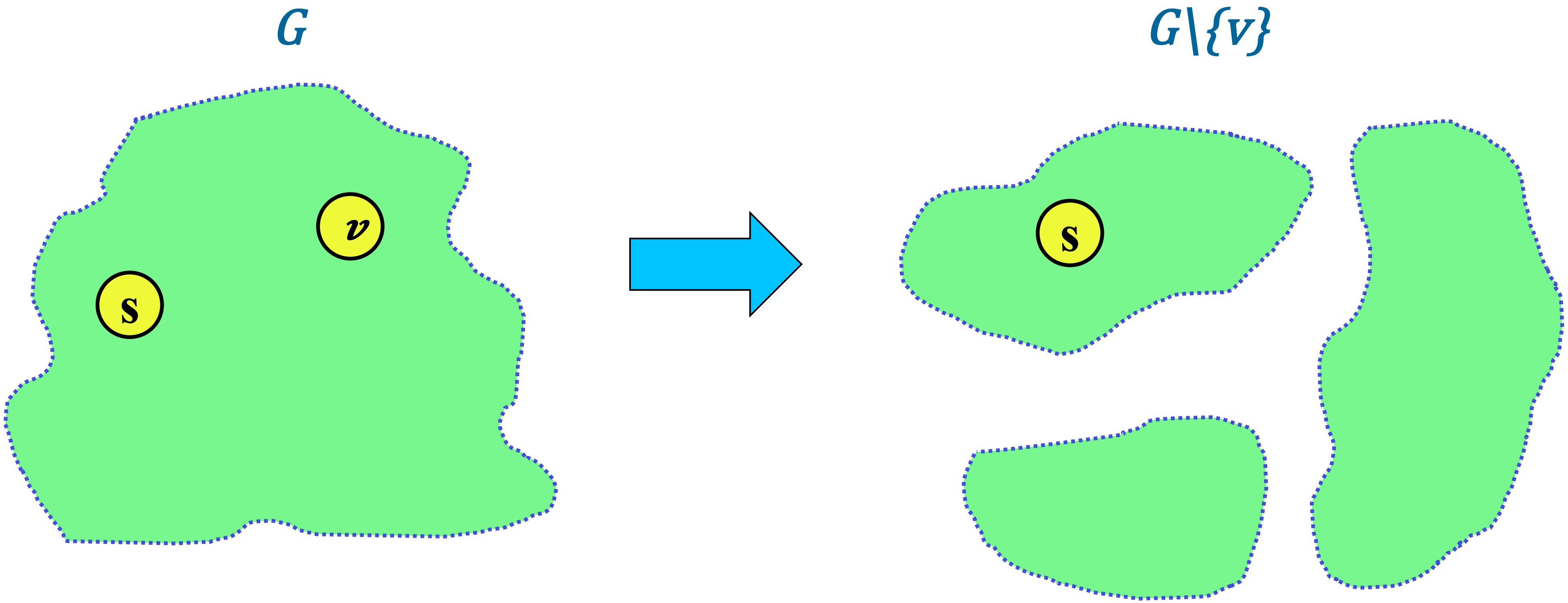
Proof**:** Let v be a strong articulation point



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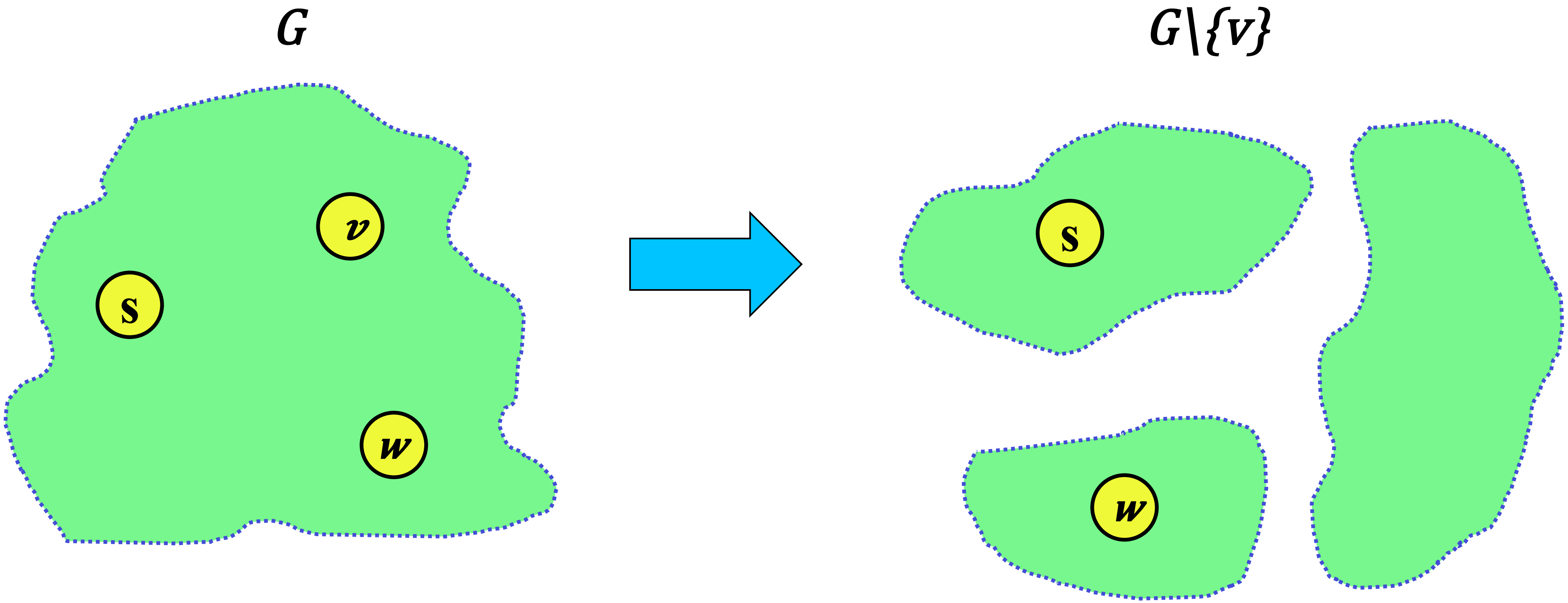

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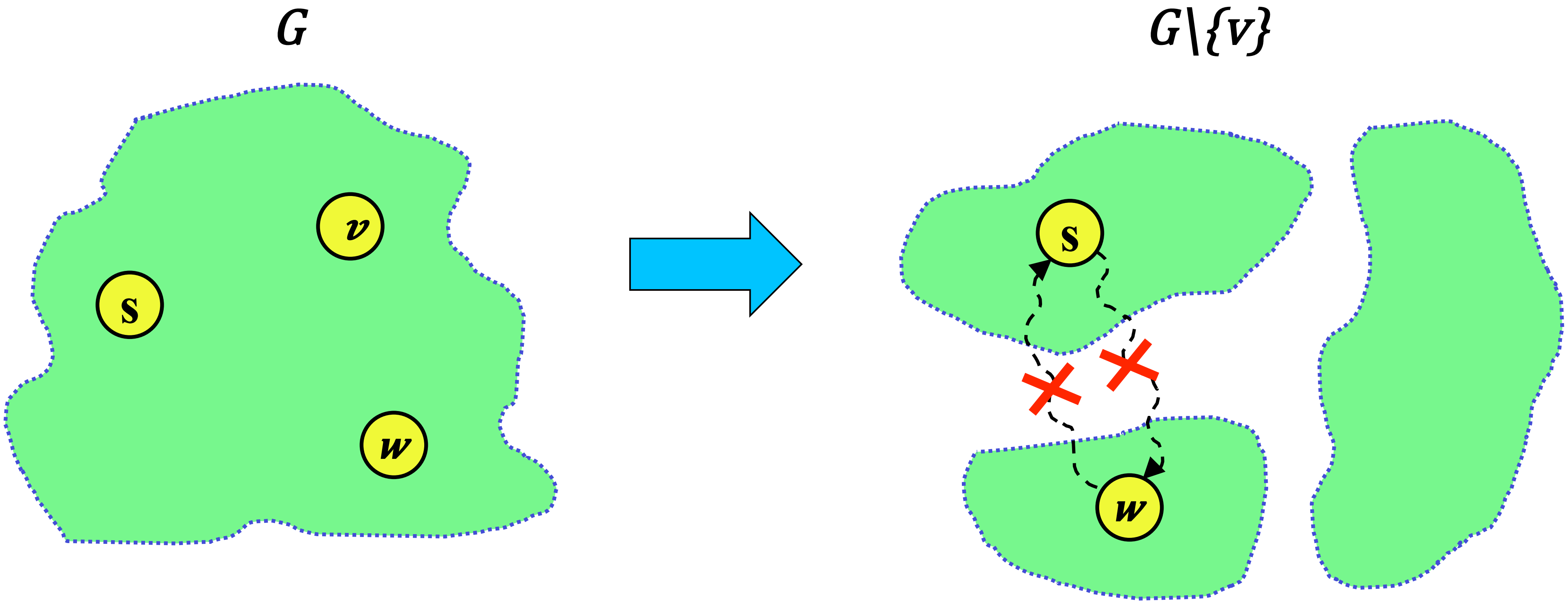

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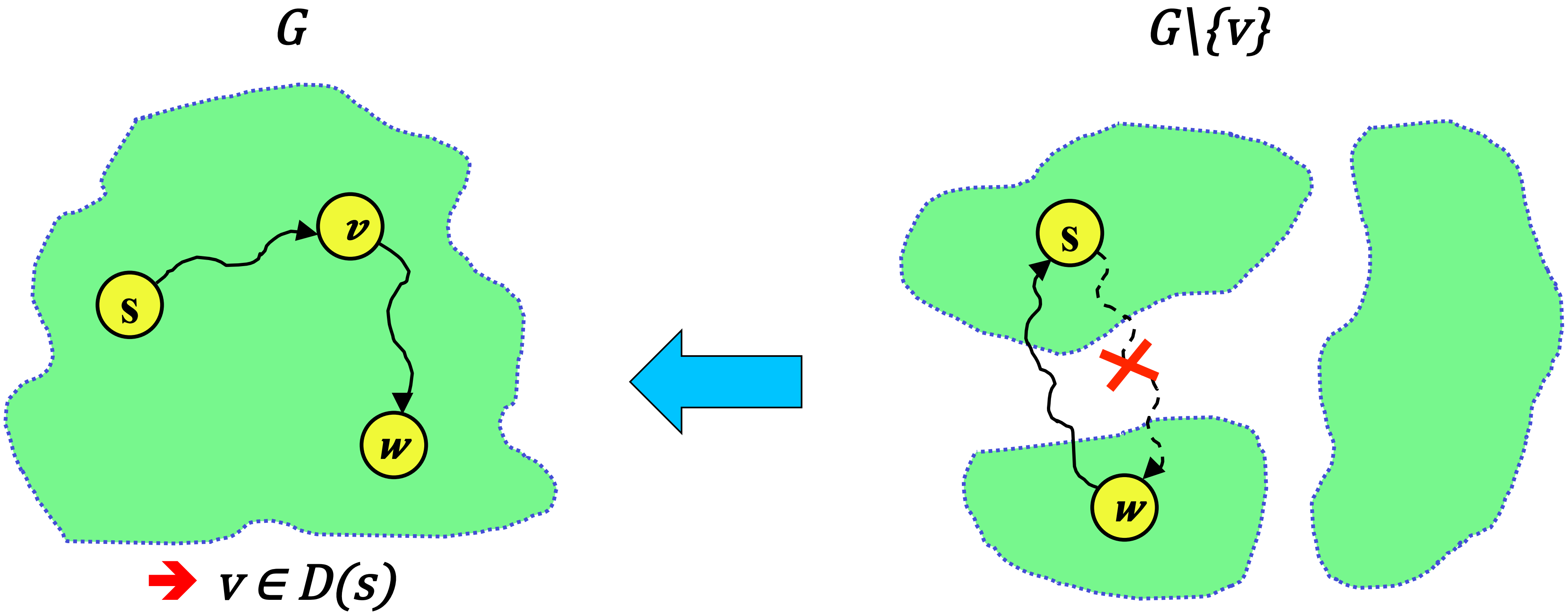

➔ $v \in D(s)$

Strong Articulation Points

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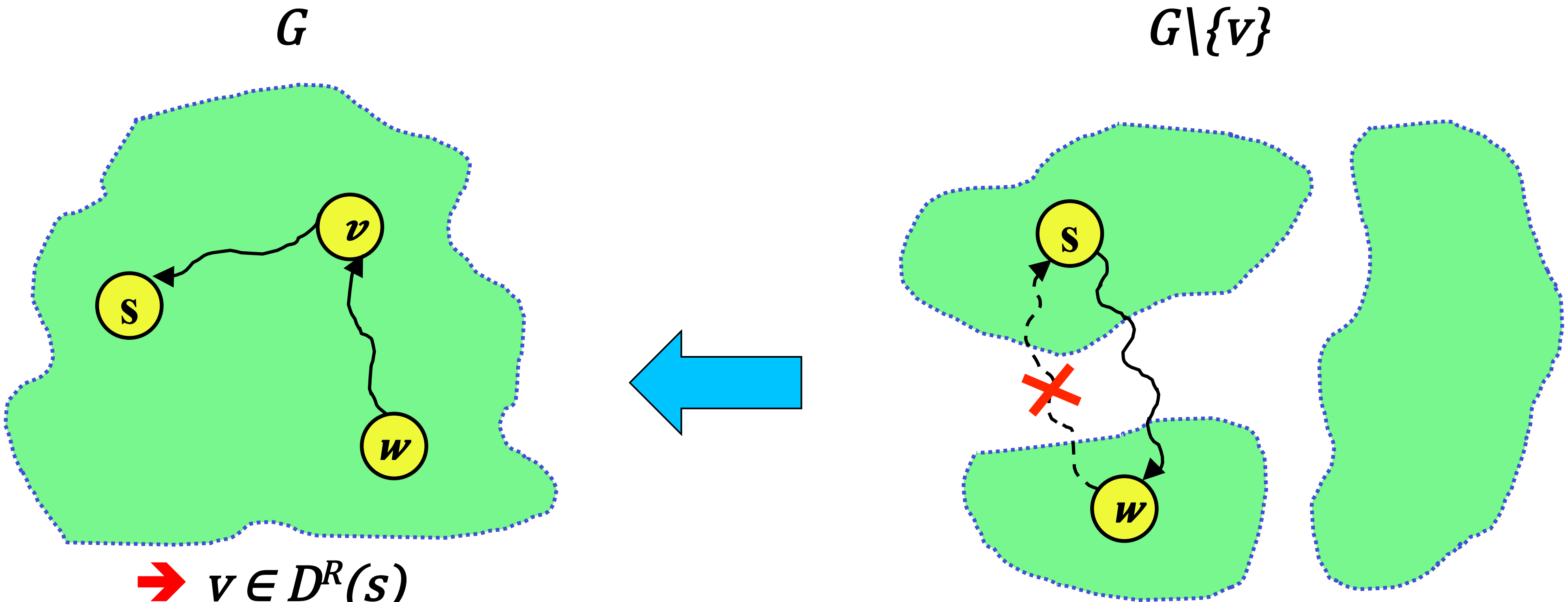


➔ $v \in D^R(s)$

Linear-Time Algorithm

Input: A strongly connected graph $G = (V, E)$, with n vertices and m edges.

Output: The strong articulation points of G.

1. Choose arbitrarily a vertex $s \in V$ in G, and test whether s is a strong articulation point in G. If s is a strong articulation point, then output s.

2. Compute and output $D(s)$, the set of non-trivial dominators in the flow graph $G(s) = (V,E,s)$.

3. Compute the reversal graph $G^R = (V,E^R)$.

4. Compute and output $D^R(s)$, the set of non-trivial dominators in the flow graph $G^R(s) = (V,E^R,s)$.

Total time is $O(m+n)$

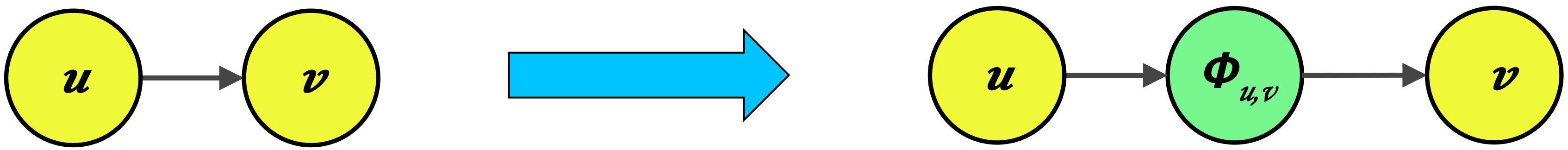
Strong Bridges

Strong Bridges

1. Reduction:

Lemma. *If there is an algorithm to compute the strong articulation points of a strongly connected graph in time $T(m,n)$, then there is algorithm to compute the strong bridges of a strongly connected graph in time $O(m + n + T(2m, n + m))$.*

"Proof" :



Mainly of theoretical interest (# vertices blows up)

Strong Bridges

2. Edge Dominators

Edge (u,v) *dominates* vertex w if every path from s to v contains edge (u,v)

If edge (u,v) dominates vertex w, and every other edge dominator of u dominates w, we say that (u,v) is an *immediate edge dominator* of vertex w.

If a vertex has an edge dominator, then it has a *unique* immediate edge dominator.

With some care, able to extend all the theory from (vertex) dominators to edge dominators.

Given a flow graph $G(s) = (V,E,s)$, edge dominators can be computed in time $O(m+n)$. But you need to re-implement code for dominators.

Edge Dominators in Practice

Lemma. [Tarjan 1974] *Let $G = (V,E,s)$ be a flow graph and let T be a DFS tree of G with start vertex s. Edge (v,w) is an edge dominator in flow graph G if and only if all of the following conditions are met:*
- (v,w) is a tree edge,
- w has no entering forward edge or cross edge, and
- there is no back edge (x,w) such that w does not dominate x.

Need to (1) compute dominator tree $DT(s)$ and (2) check whether w ancestor of x in $DT(s)$ for back edge (x,w).

Given a flow graph $G(s) = (V,E,s)$, edge dominators can be computed in time $O(m+n)$. Reuse code for (vertex) dominators. More efficient in practice. But still slightly slower than (vertex) dominators.

Computing All Strong Bridges

Given a strongly connected graph $G=(V,E)$, let

- $G(s) = (V,E,s)$ be the flow graph with start vertex s
- $ED(s)$ the set of *edge dominators* in $G(s)$
- $G^R(s) = (V,E^R,s)$ be the flow graph with start vertex s
- $ED^R(s)$ the set of *edge dominators* in $G^R(s)$

Theorem. *Let $G = (V,E)$ be a strongly connected graph, and let $s \in V$ be any vertex in G. Then edge (u,v) is a strong bridge in G if and only if $(u,v) \in ED(s) \cup ED^R(s)$.*

Incidentally, this proves also that can be at most $2n-2$ strong bridges in a directed graph.

Today's Outline

1. 2-Connectivity on directed graphs
2. Algorithms for strong articulation points and strong bridges
3. **Experiments (very rough, still ongoing)**
4. Open Problems

2-Connectivity

Can 2-connectivity be useful to understand the (macroscopic) structure of social networks / web graphs / other networks?

Do social networks / web graphs / other networks have different 2-connectivity properties?

Nodes / links which act more as "information" gateways (tweets / diseases / etc…) in the network?

2-Vertex Cores

Delete recursively all the strong articulation points of a directed graph *G*, as follows

While there are strong articulation points in G do:

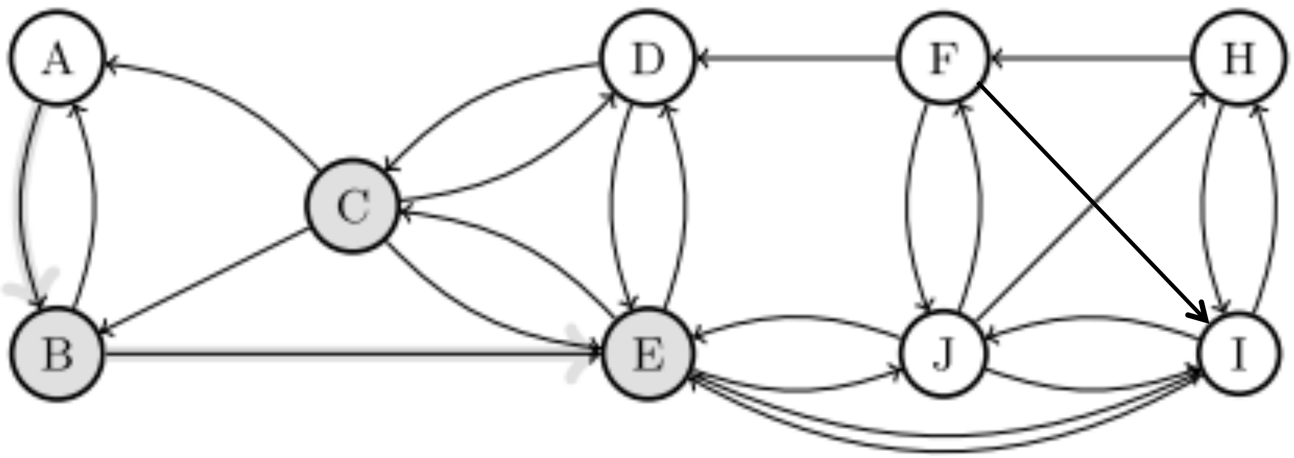
1. Let G' be the graph defined by all the s.c.c.'s of G;
2. Set G to be the graph obtained by deleting all the strong articulation points in G' together with their incident edges.

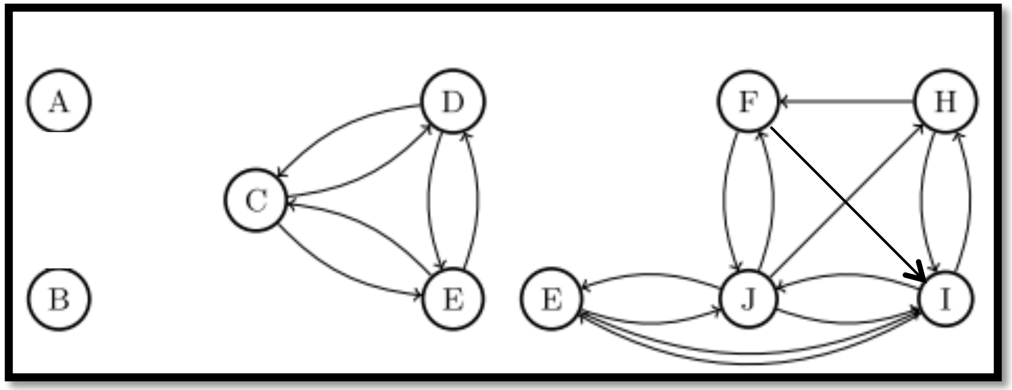
Let G_f be the final graph obtained at the end of this process.

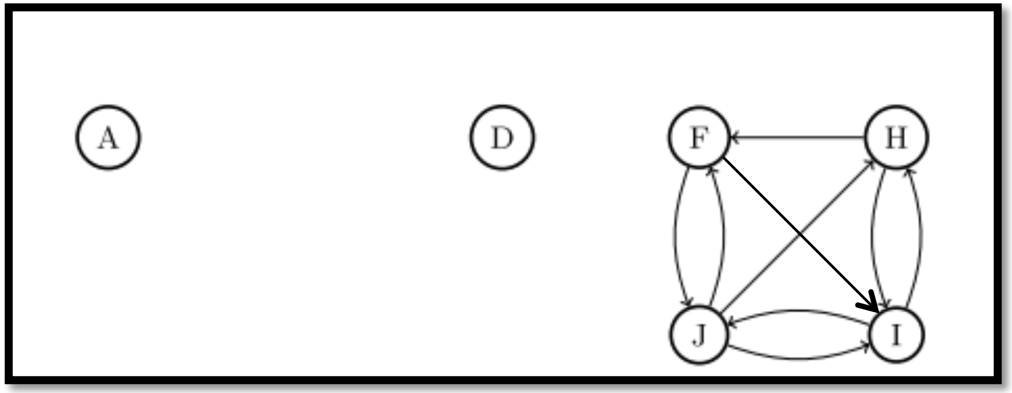
We call the s.c.c.'s of G_f the ***2-vertex connectivity cores*** of the original graph G

2-vertex connectivity cores are subsets of 2-vertex-connected components

2-Vertex-Connected Components and 2-Vertex Cores







2-Edge Cores

Delete recursively all the strong bridges of directed graph *G*, as follows

While there are strong bridges in G:

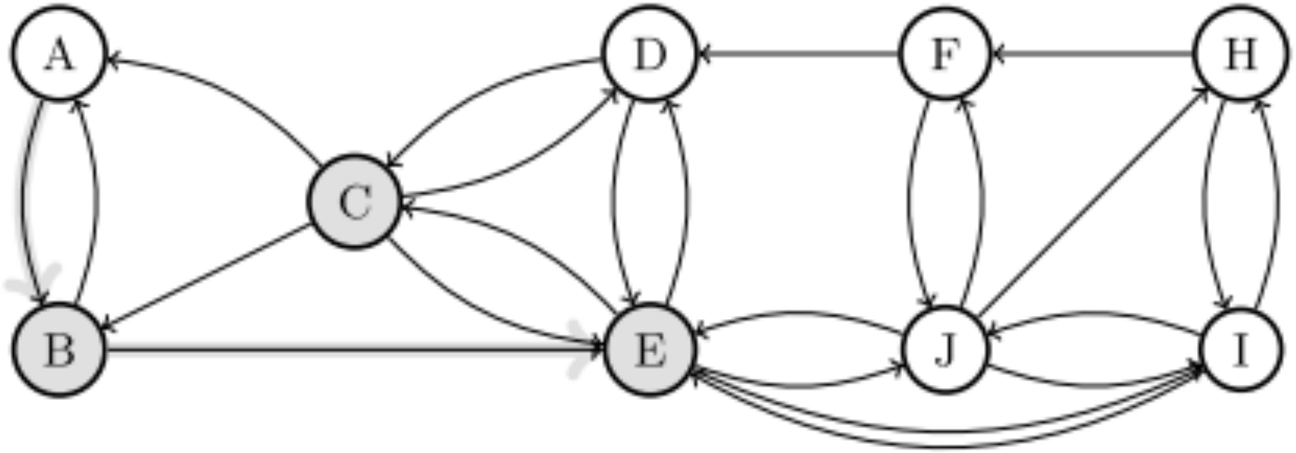
1. Let G' be the graph defined by all the s.c.c.'s of G;
2. Set G to be the graph obtained by deleting all the strong bridges in G'.

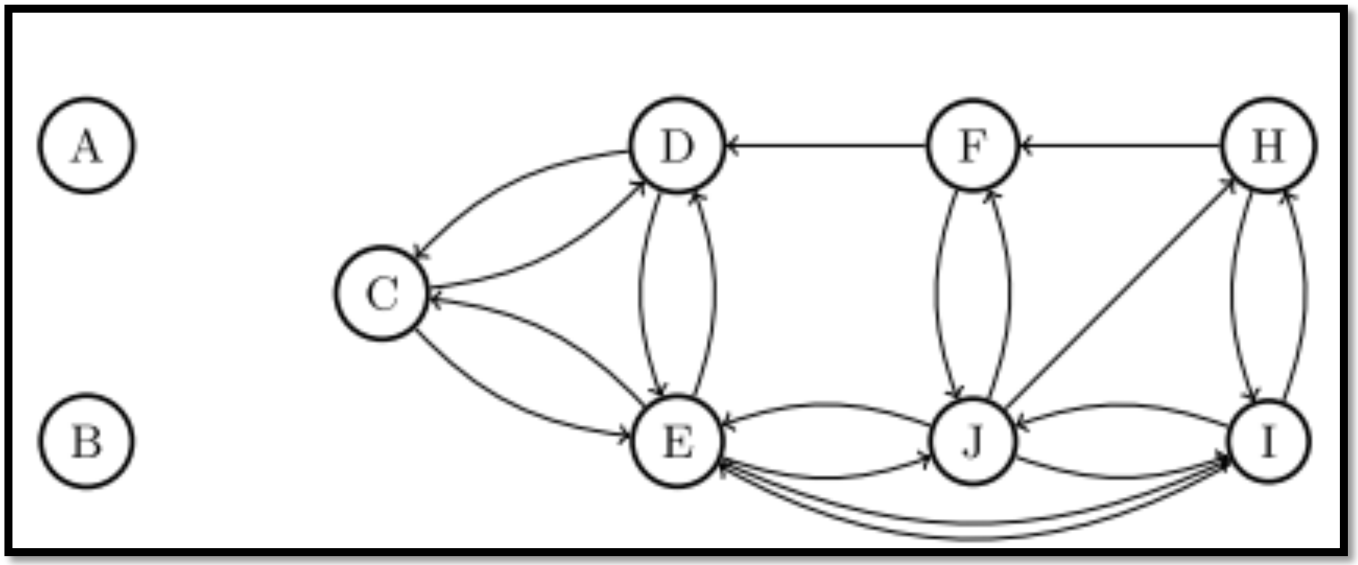
Let G_f be the final graph obtained at the end of this process.

The s.c.c.'s of G_f are the ***2-edge connectivity cores*** of the original graph G

2-edge connectivity cores are exactly 2-edge-connected components

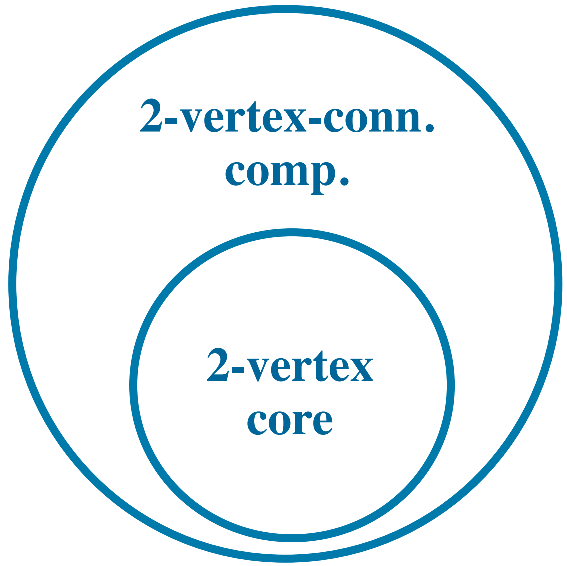
2-Edge-Connected Components = 2-Edge Cores





A Hierarchy of 2-Components

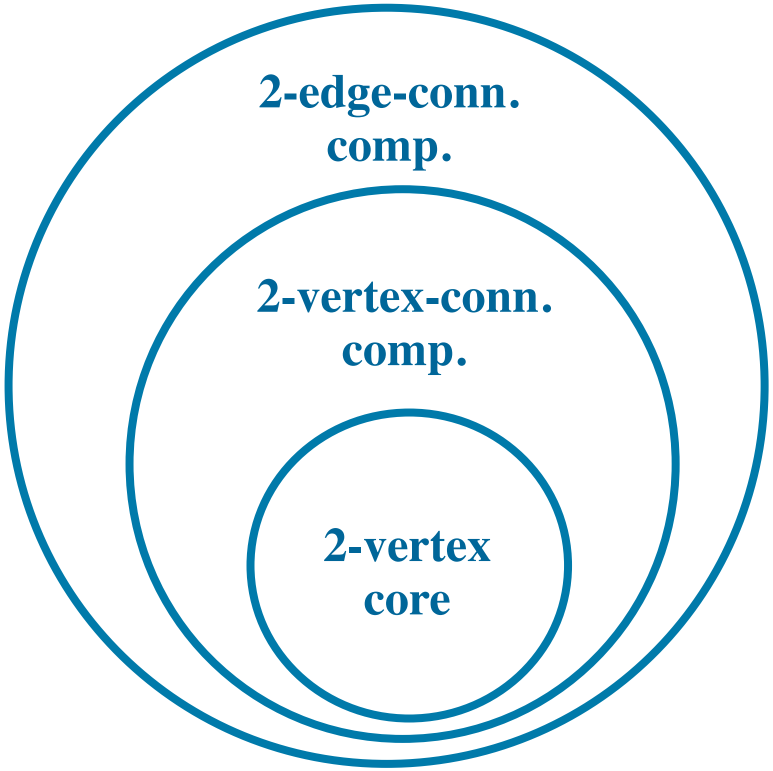
2-vertex core subset of 2-vertex-connected component (modulo degenerate components)



A Hierarchy of 2-Components

2-vertex core subset of 2-vertex-connected component (modulo degenerate components)

2-vertex-connected component subset of 2-edge-connected component



A Hierarchy of 2-Components

2-vertex core subset of 2-vertex-connected component (modulo degenerate components)

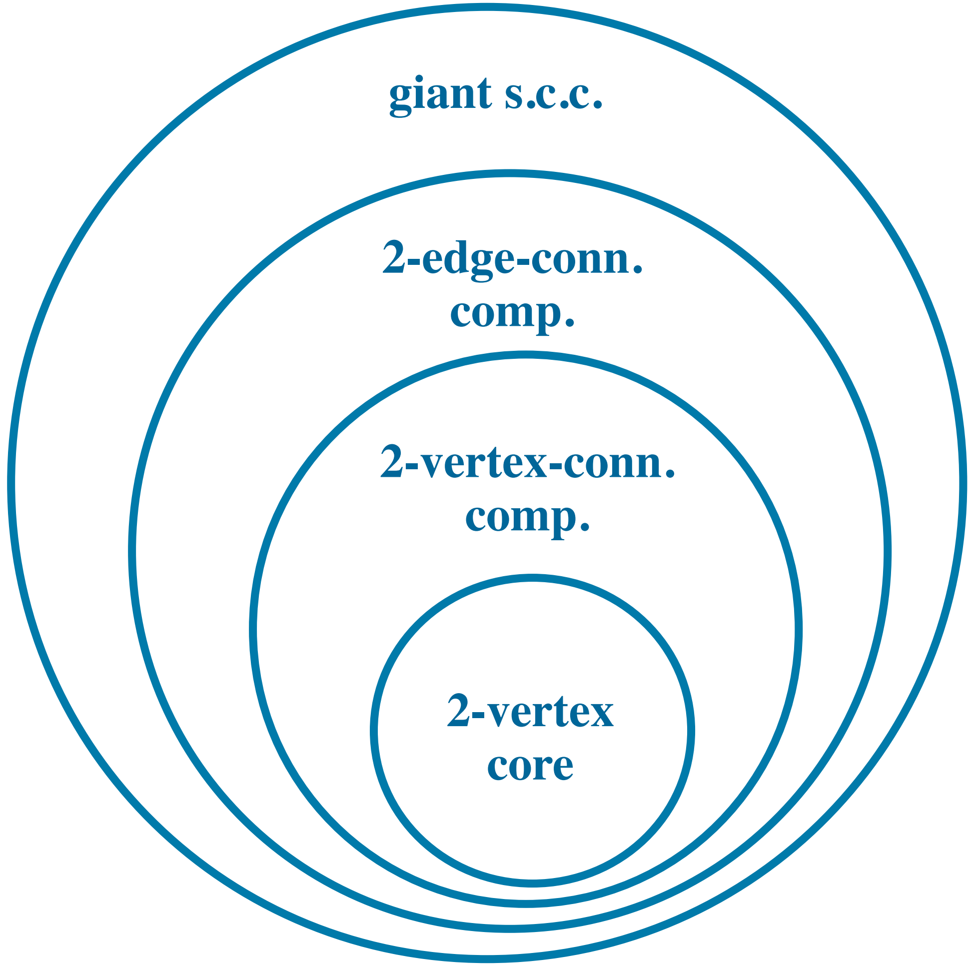
2-vertex-connected component subset of 2-edge-connected component

2-edge-connected component subset of strongly connected component

Roughly speaking:

2VC $\subseteq$ 2VCC $\subseteq$ 2ECC $\subseteq$ SCC

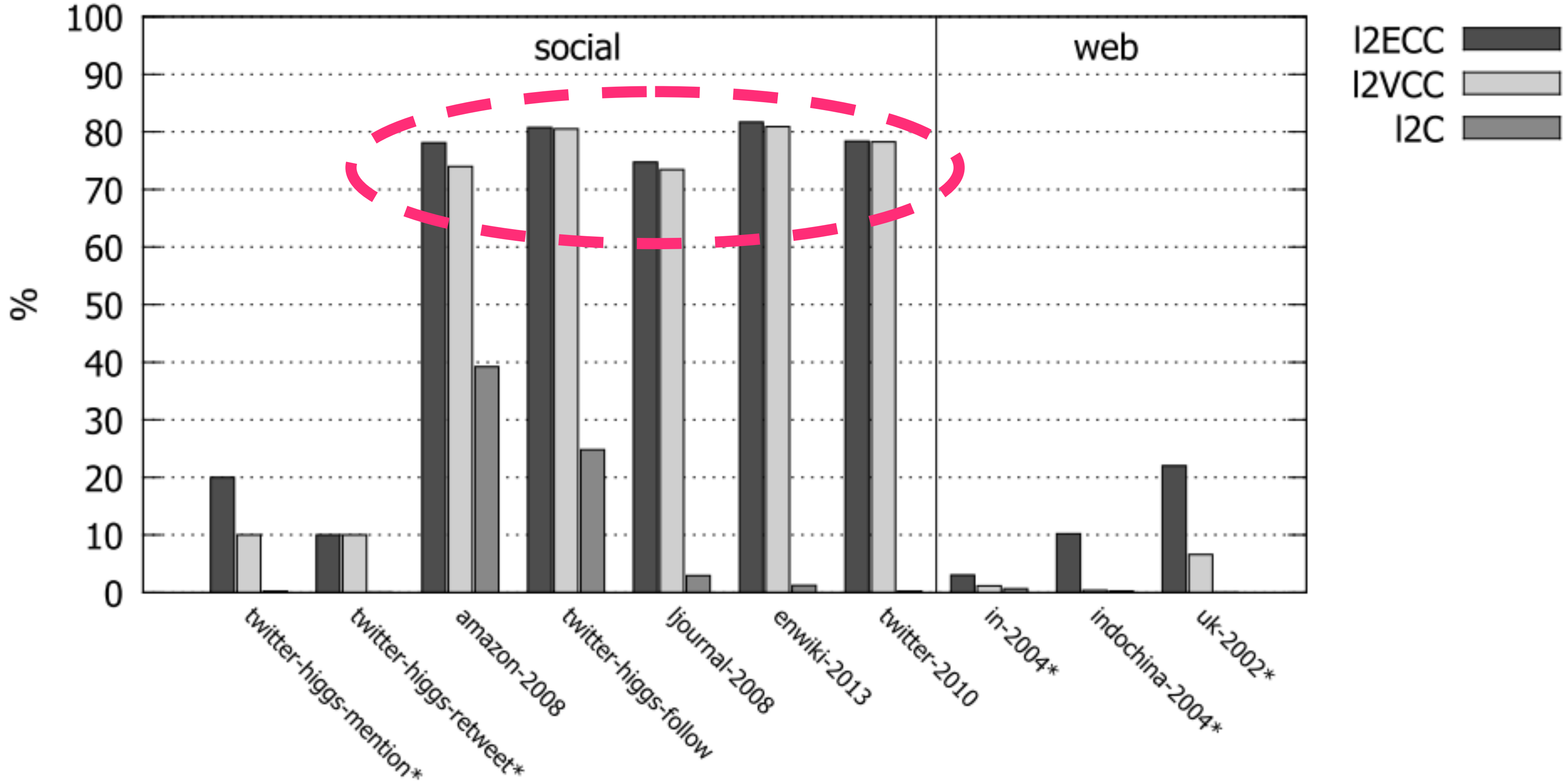
(Will integrate blocks in our experiments soon)



Data Set

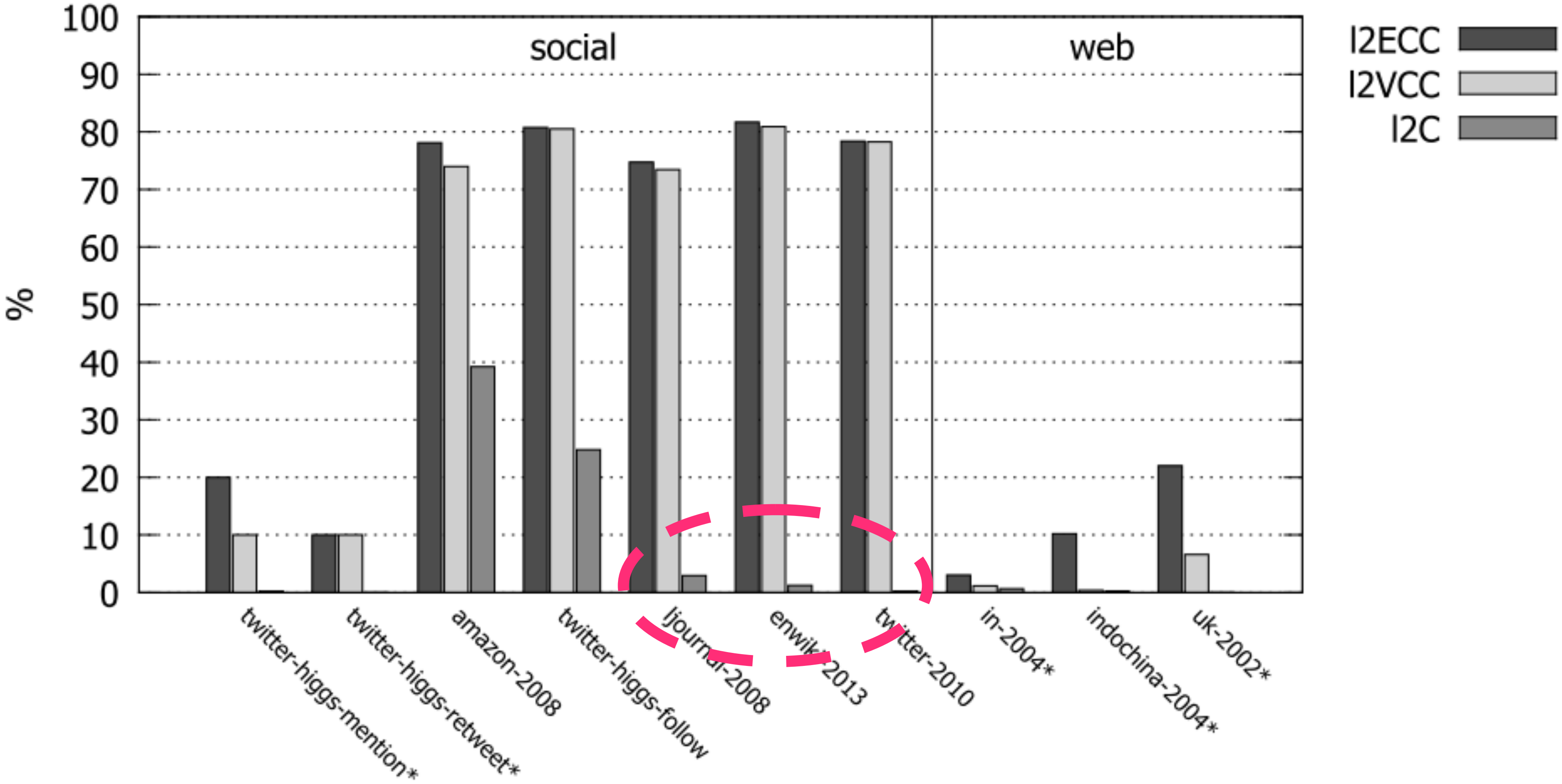
Graph	Type	Repository	n	m	lSCC
twitter-higgs-mention	Social	SNAP	303 K	448.8 K	3%
twitter-higgs-retweet	Social	SNAP	427 K	733.6 K	2%
amazon-2008	Social	WebGraph	735 K	5.1 M	85%
twitter-higgs-follow	Social	SNAP	456 K	14.8 M	79%
ljournal-2008	Social	WebGraph	5.4 M	79 M	78%
enwiki-2013	Social	WebGraph	4.3 M	101.3 M	89%
twitter-2010	Social	WebGraph	41.6 M	1.5 G	80%
in-2004	Web	WebGraph	1.4 M	17 M	43%
indochina-2004	Web	WebGraph	7.4 M	194.1 M	51%
uk-2002	Web	WebGraph	18.5 M	298.1 M	65%

Social: Bigger 2-Components…



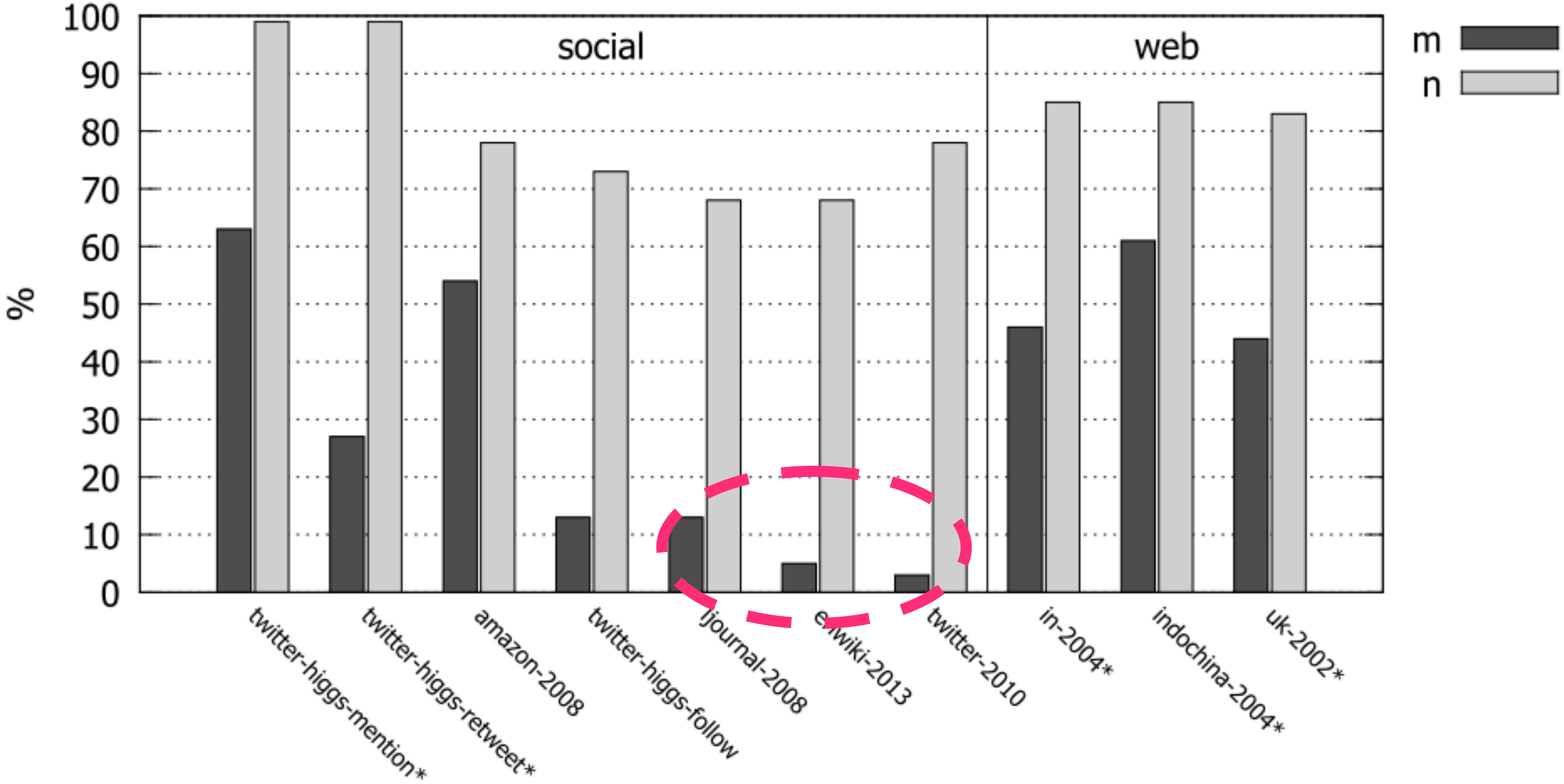
Size of the giant 2-vertex- and 2-edge-connected component (I2VCC and I2ECC) and in the largest 2-vertex-connected core (I2C).
(Expressed as % of vertices in the ISCC)

…some have small cores…



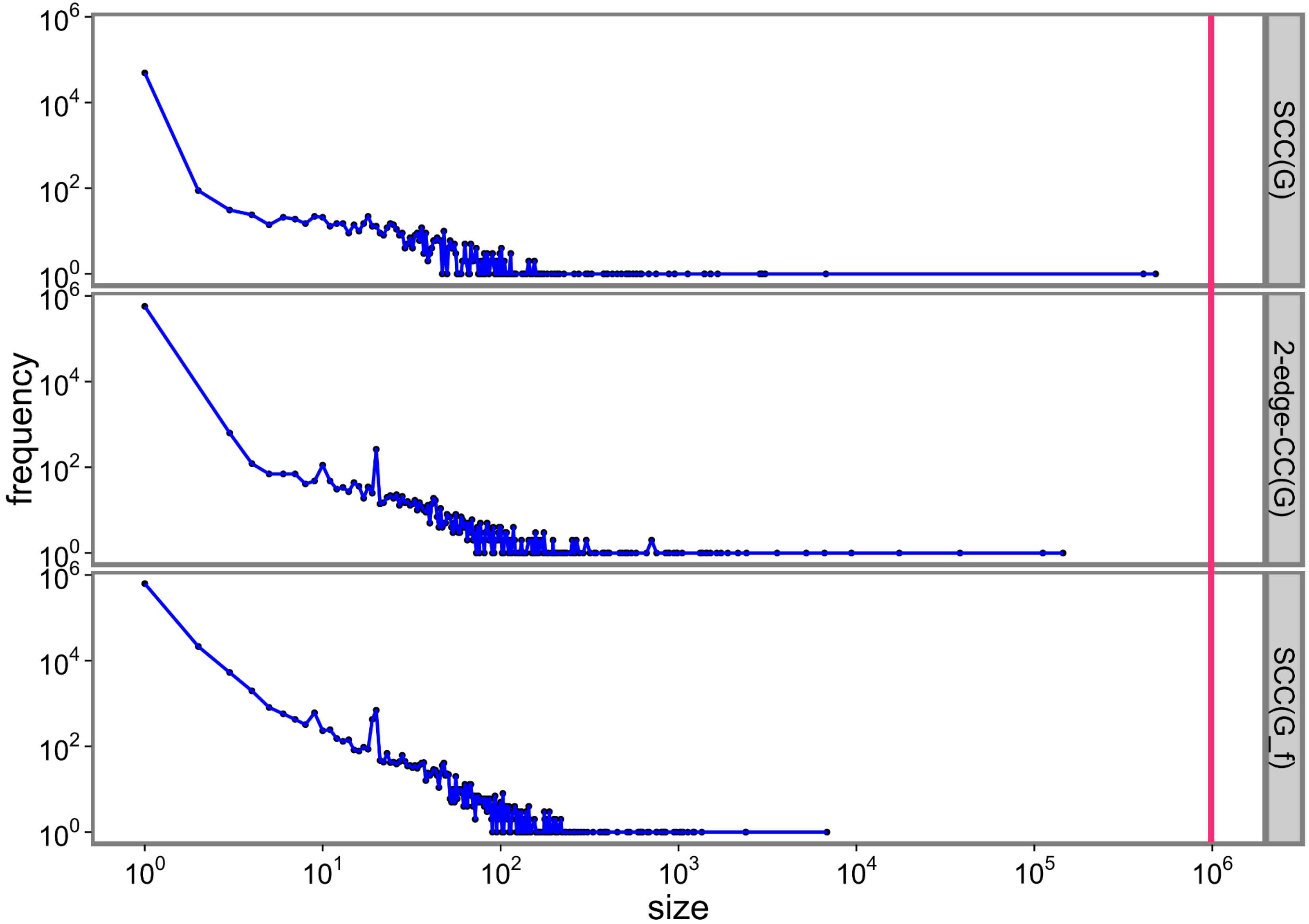
Size of the giant 2-vertex- and 2-edge-connected component (I2VCC and I2ECC) and in the largest 2-vertex-connected core (I2C).
(Expressed as % of vertices in the ISCC)

…but are buried deep inside



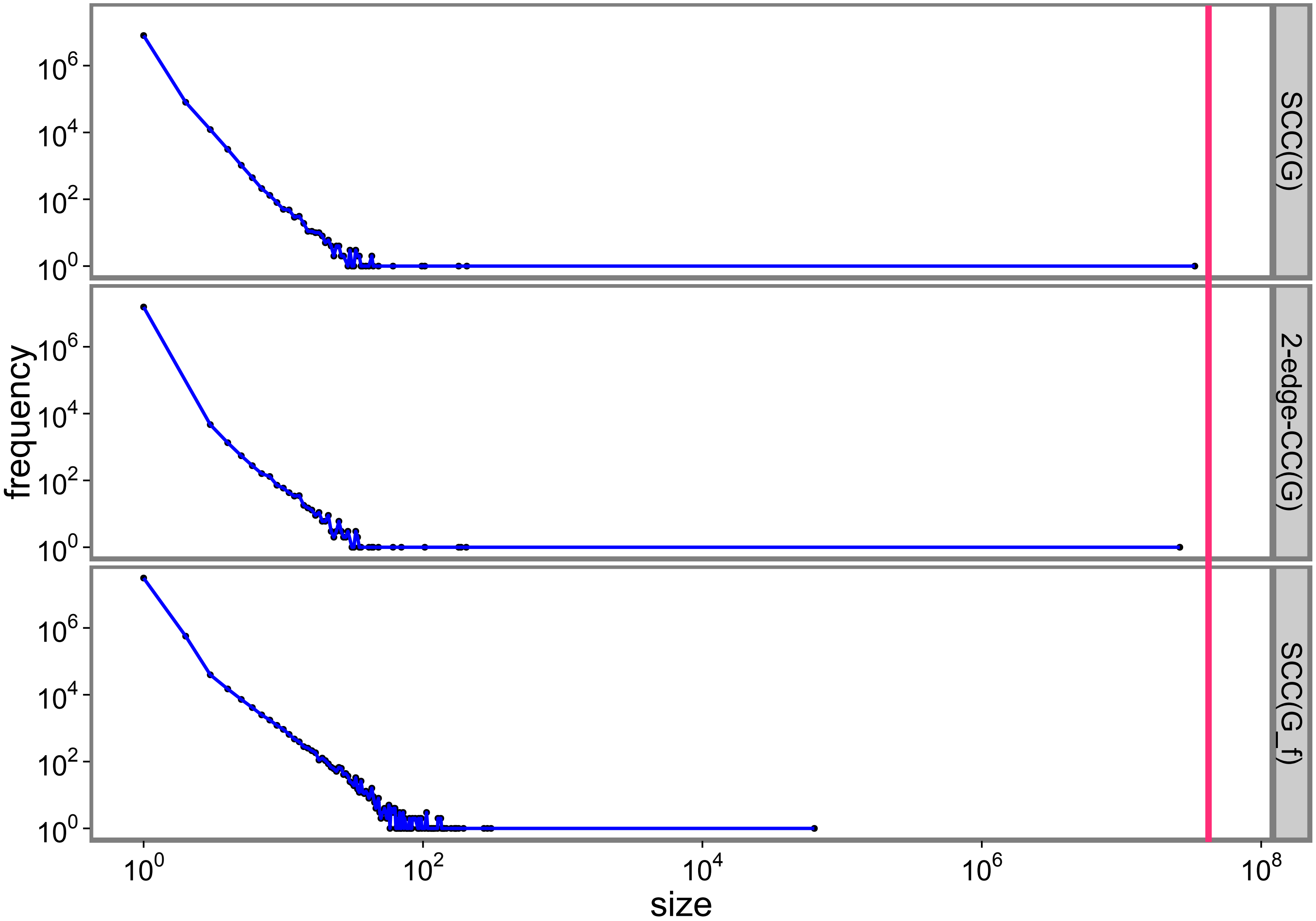
Number of vertices and edges in the final graph G_f obtained after removing recursively all strong articulation points.
(Expressed as % of n and m)

Biconnectivity: Web Graphs



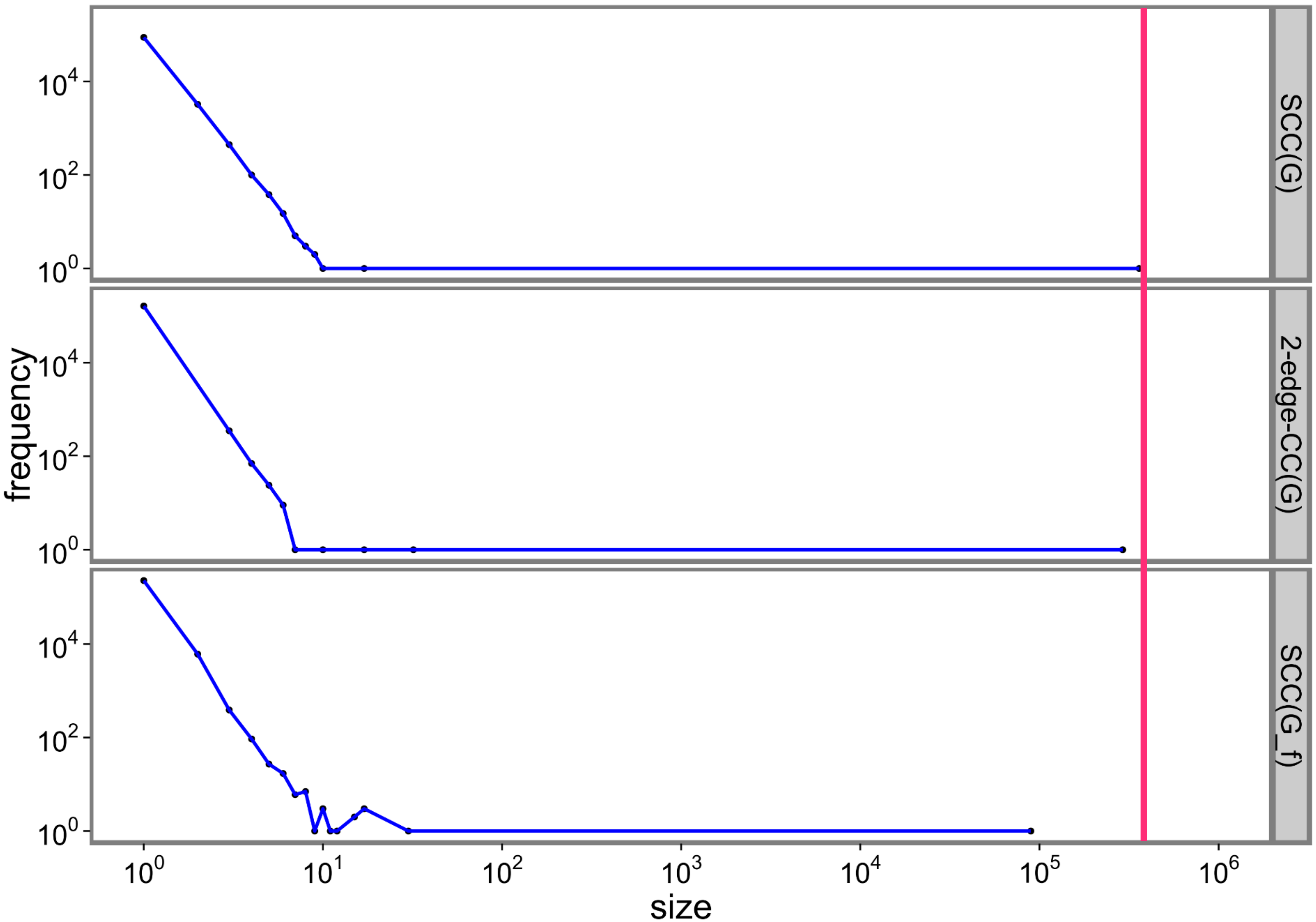
uk-2002@1M (n = 1M, m = 41.2M)

Biconnectivity: Social Networks



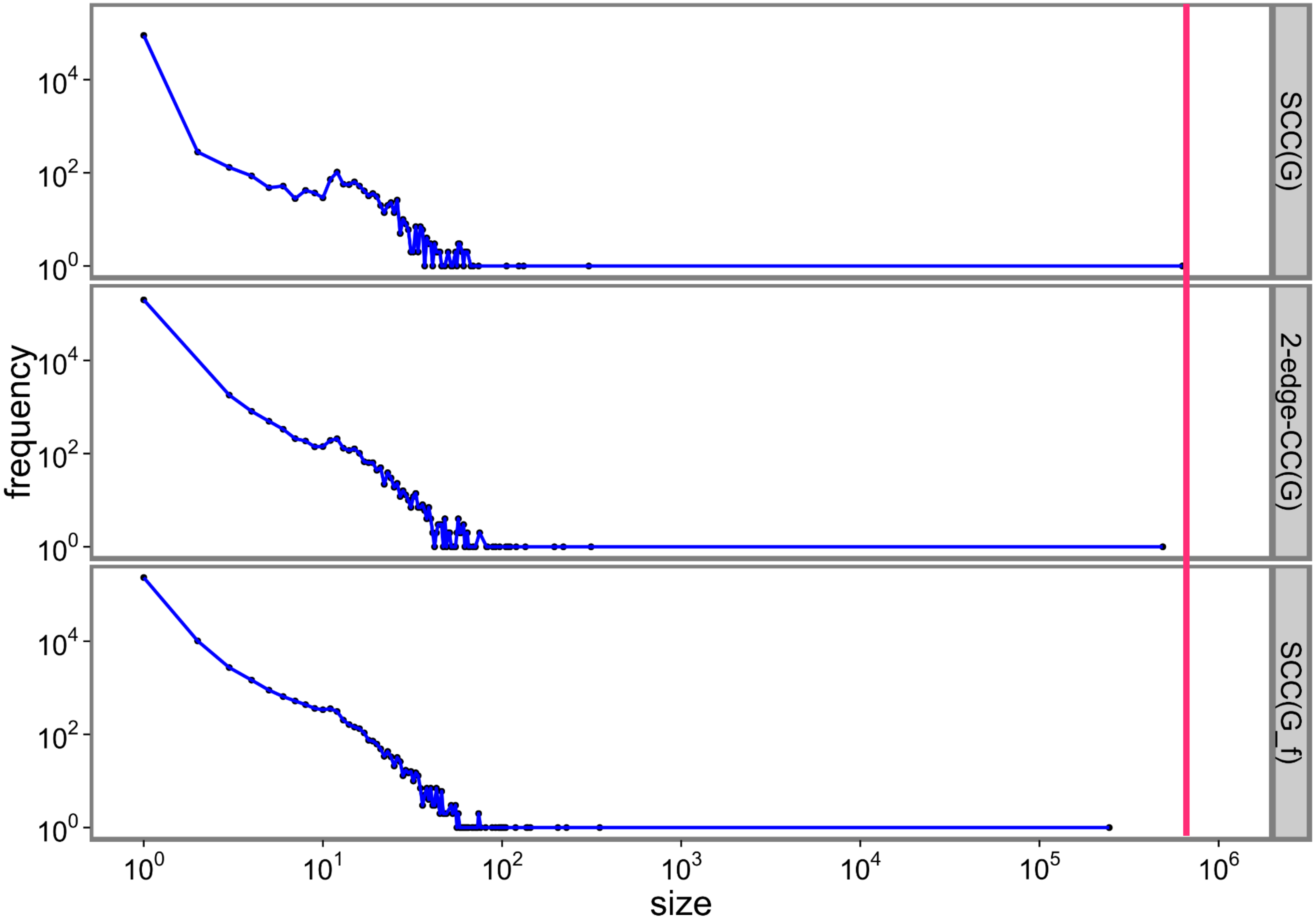
twitter-2010 (n = 41.6M, m = 1,468.3M)

Biconnectivity: Social (sub)networks



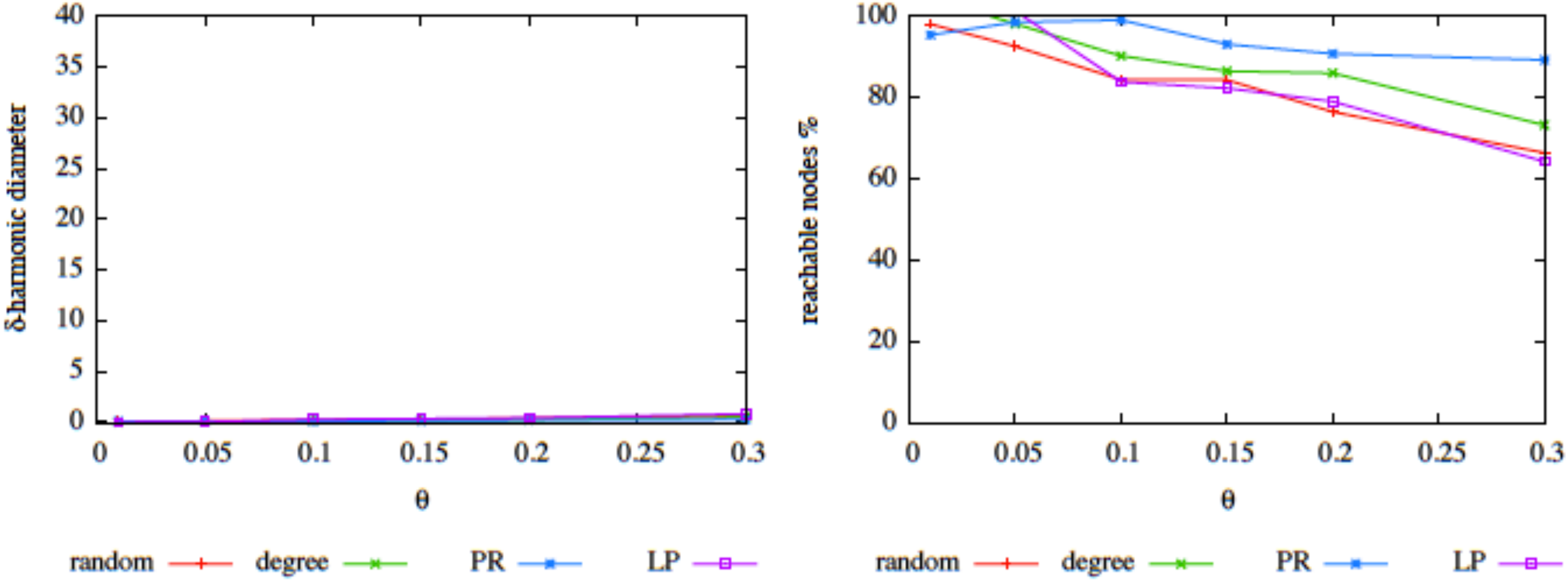
twitter-higgs-follow (n = 456K, m = 14.8M)

Biconnectivity: Co-Purchase

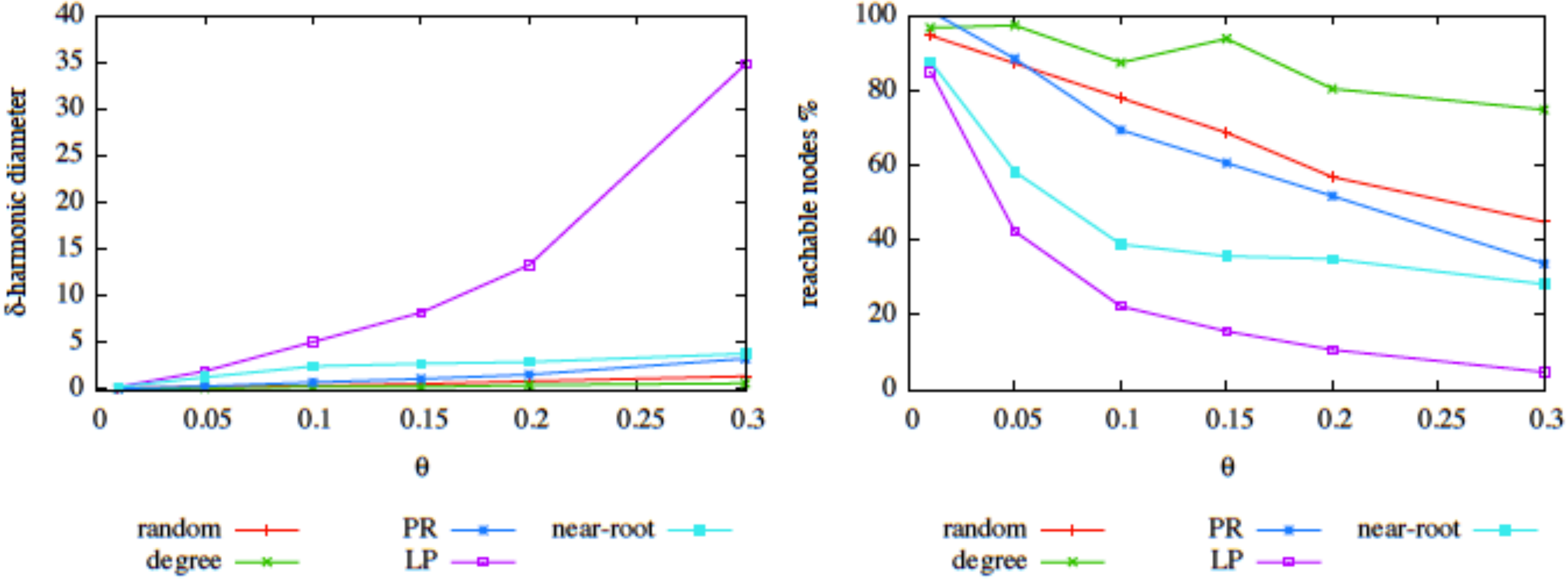


amazon-2008 (n = 735K, m = 5.1M)

[Boldi, Rosa & Vigna 2013]

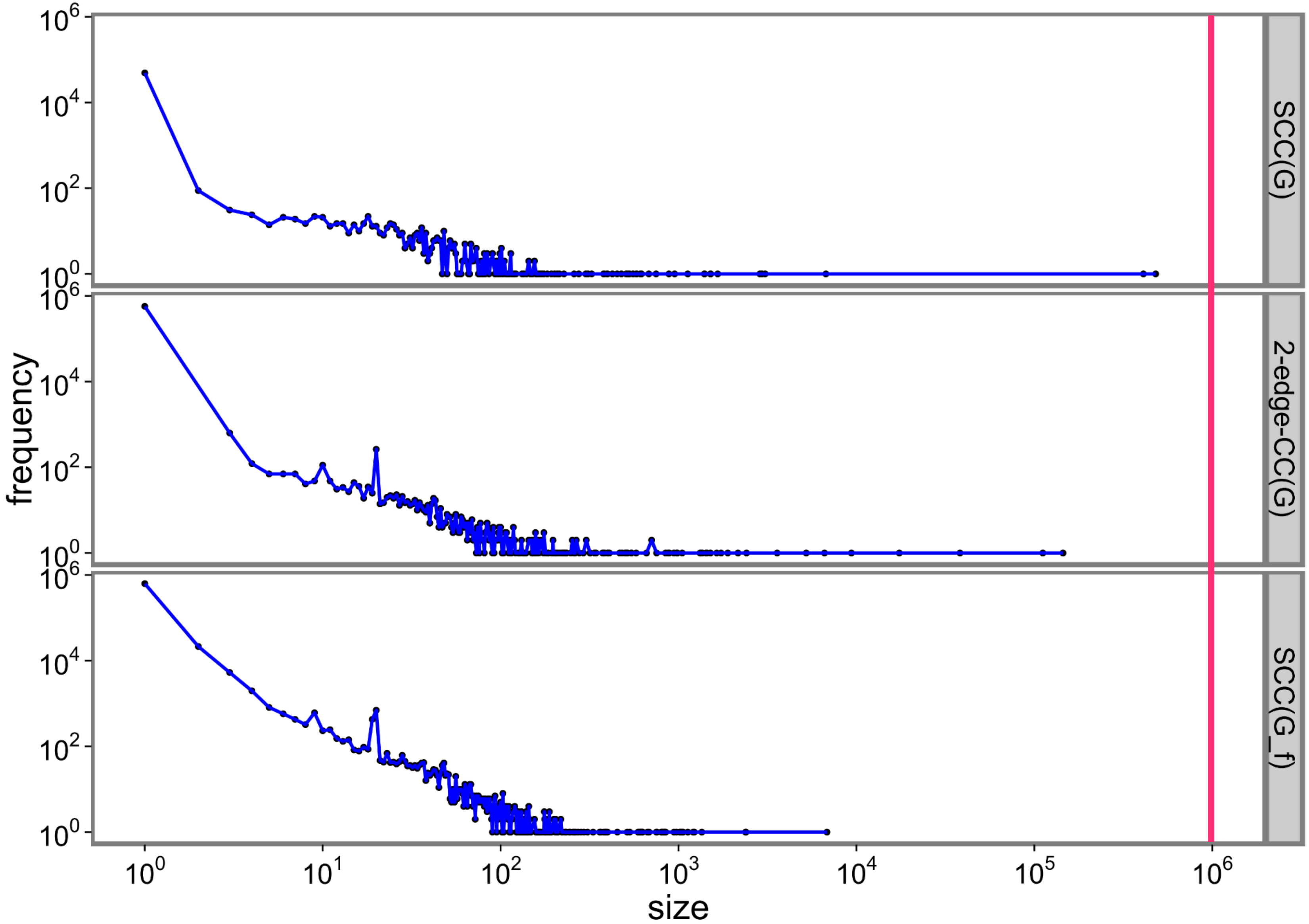


Amazon



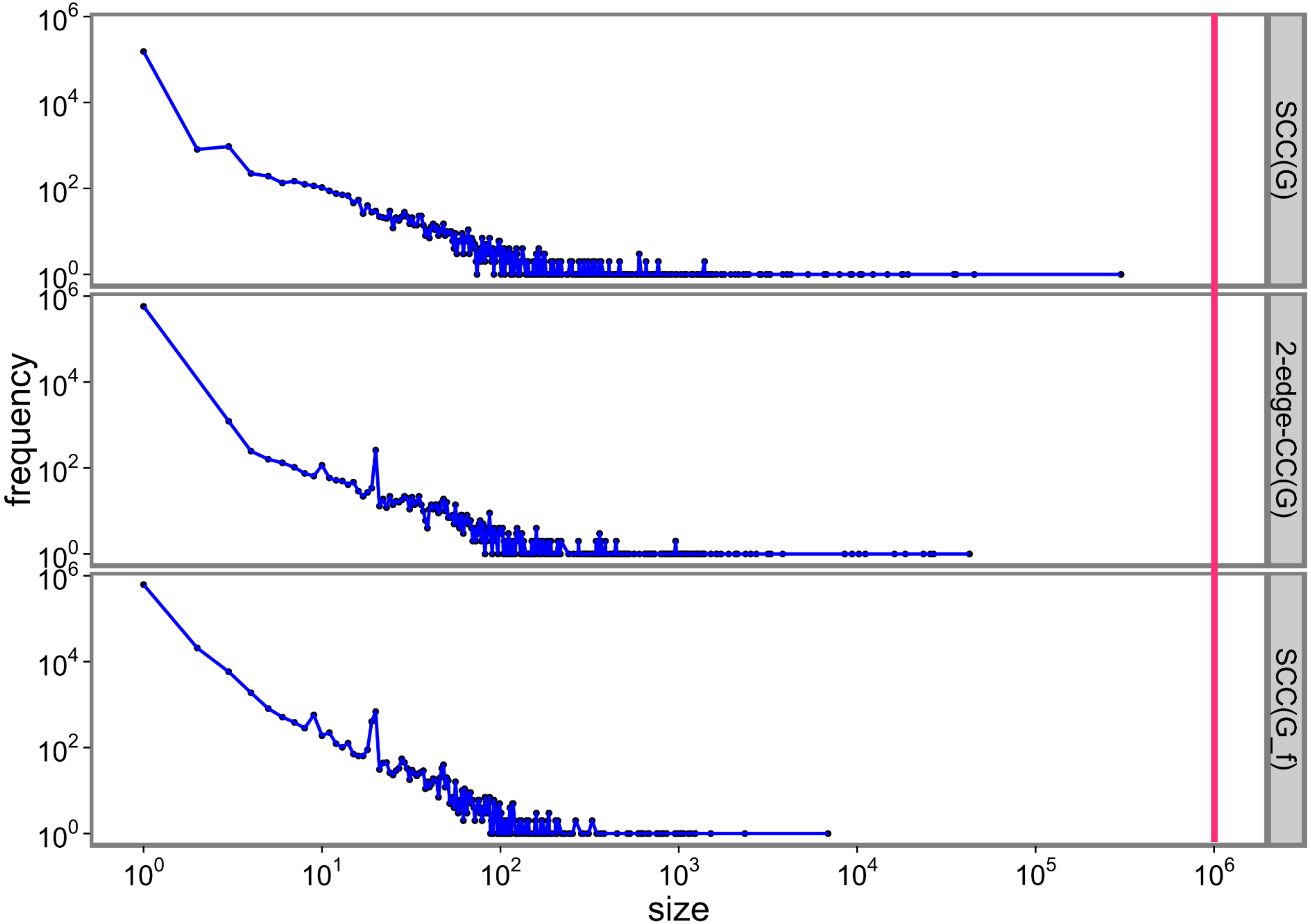
.it

30% LP on Web Graphs



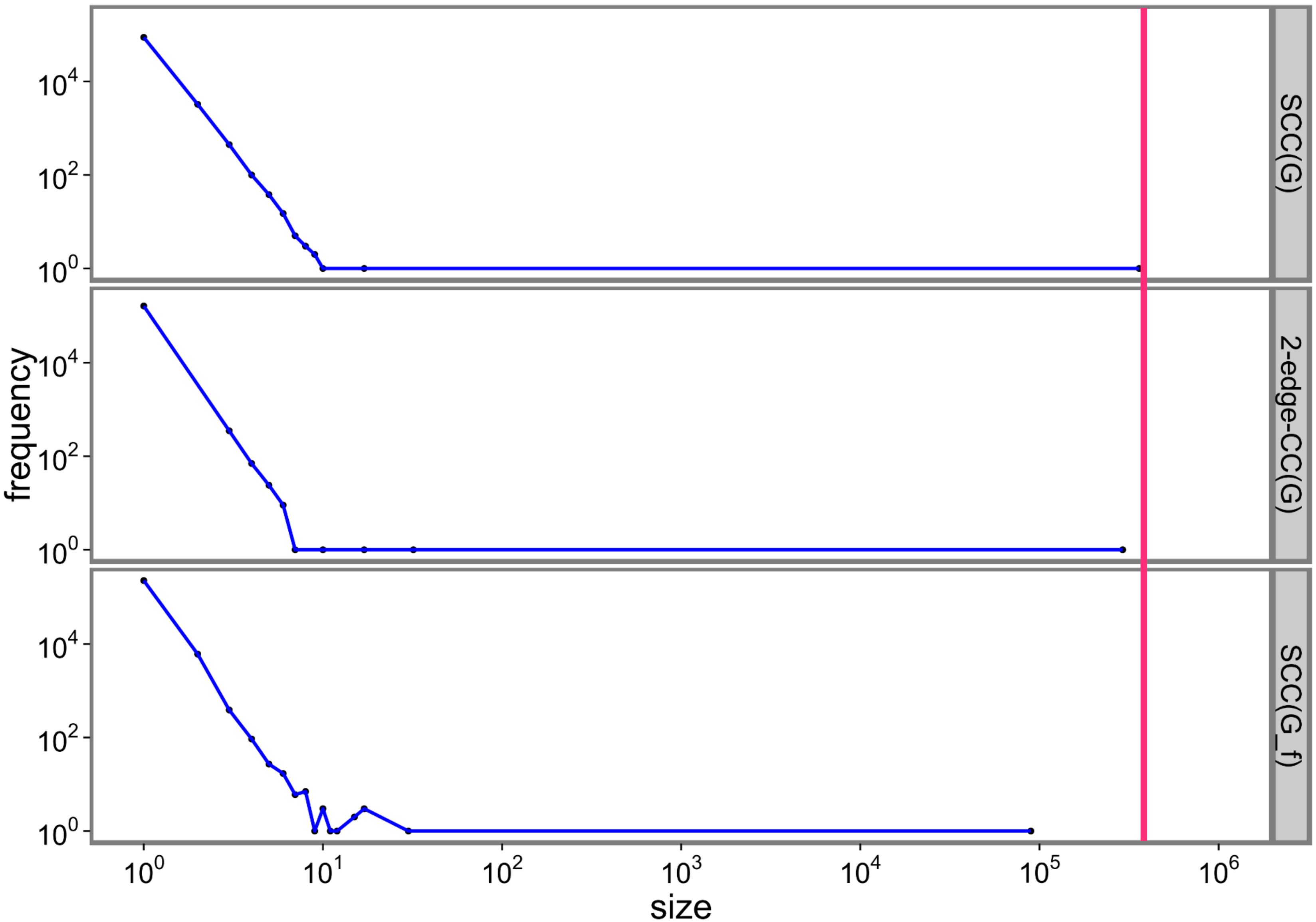
uk-2002@1M (n = 1M, m = 41.2M)

30% LP on Web Graphs



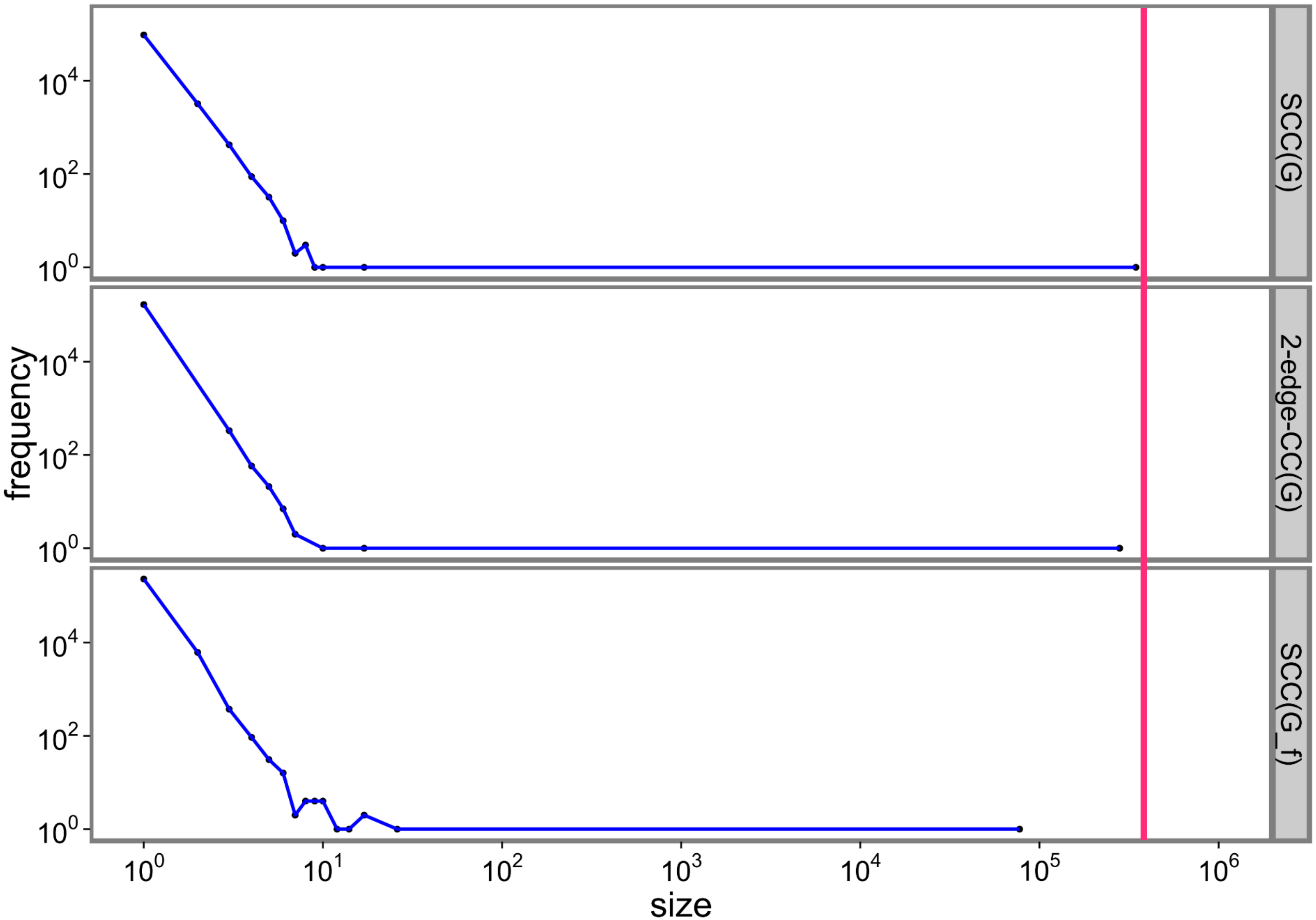
uk-2002@1M (n = 1M, m = 41.2M)

30% LP on Social Networks



twitter-higgs-follow (n = 456K, m = 14.8M)

30% LP on Social Networks



twitter-higgs-follow (n = 456K, m = 14.8M)

Today's Outline

1. 2-Connectivity on directed graphs
2. Algorithms for strong articulation points and strong bridges
3. Experiments
4. **Open Problems**

Open Problems

Can the 2-vertex and 2-edge-connected components of a directed graph be computed in linear time?

> Best known time is $O(n(m+n))$ by repeatedly deleting strong articulation points / strong bridges.
>
> Can a dynamic algorithm help you?

Higher connectivity cuts in strongly connected graphs in linear time? (e.g., separation pairs: vertex and edge cuts of cardinality 2)

Open Problems / Future Work

Would like to understand more the structure and the properties of 2-connectivity components / blocks (especially in real-world graphs)

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