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A note on the complexity of minimum dominating set

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Abstract

The currently (asymptotically) fastest algorithm for minimum dominating set on graphs of n nodes is the trivial $\Omega(2^n)$ algorithm which enumerates and checks all the subsets of nodes. In this paper we present a simple algorithm which solves this problem in $O(1.81^n)$ time.

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1. Introduction

Since the seminal work of Tarjan and Trojanowski [16], a lot of effort has been devoted to develop faster and faster (exponential-time) exact algorithms for NP-hard and NP-complete problems, such as *maximum independent set* [3,10,14,15], *vertex cover* [1,5,12], *(maximum) satisfiability* [2,7,11,13], *3-coloring* [4,8] and many others.

Minimum dominating set is one of the most basic NP-hard problems [9]. The currently (asymptotically) fastest algorithm to solve this problem on graphs of n nodes is the trivial $\Omega(2^n)$ algorithm which enumerates and checks all the subsets of nodes.

In this paper we present a $O(1.3424^k)$ algorithm for *minimum set cover*, where k , the *dimension* of the problem, is the sum of the number of sets available and of the number of elements which need to be covered. Minimum dominating set can be formulated as a

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1 minimum set cover problem of dimension $k = 2n$. It follows that minimum dominating set 1
2 can be solved in $O(1.3424^{2n}) = O(1.8021^n)$ time. 2

3 1.1. Preliminaries 3

4 We use standard set notation. Let \mathcal{S} be a collection of subsets of a given universe \mathcal{U} . For 4
5 the sake of simplicity, we assume that \mathcal{S} covers \mathcal{U} : 5

$$6 \quad \mathcal{U} = \mathcal{U}_{\mathcal{S}} = \bigcup_{S \in \mathcal{S}} S. \quad 6$$

7 A set cover of \mathcal{S} is a subset \mathcal{S}' of \mathcal{S} which covers \mathcal{U} . The minimum set cover problem 7
8 consists in determining the minimum cardinality $msc(\mathcal{S})$ of a set cover of \mathcal{S} . Without loss 8
9 of generality, we can assume that \mathcal{S} does not contain the empty set (since it does not 9
10 belong to any minimum set cover). We call dimension k of \mathcal{S} the sum of the cardinalities 10
11 of \mathcal{S} and \mathcal{U} : 11

$$12 \quad k = |\mathcal{S}| + |\mathcal{U}|. \quad 12$$

13 Let R be a subset of \mathcal{U} . By $del(\mathcal{S}, R)$ we denote the collection which is obtained from \mathcal{S} 13
14 by removing the elements of R from each S in \mathcal{S} , and by eventually removing the empty 14
15 sets obtained: 15

$$16 \quad del(\mathcal{S}, R) = \{S' \neq \emptyset : S' = S \setminus R, S \in \mathcal{S}\}. \quad 16$$

17 We use standard graph notation as for example in [6]. In particular, by $G = (V, E)$ we 17
18 denote a (undirected) graph, where V is the set of nodes and E is the set of edges (pairs 18
19 of distinct nodes). Two nodes u and v are adjacent if there is an edge $\{u, v\} \in E$. The 19
20 neighborhood $N(v)$ of node v is the set of nodes adjacent to v . A dominating set of G is a 20
21 subset V' of V such that every node $u \in V \setminus V'$ is adjacent to at least one node $v \in V'$, i.e.: 21

$$22 \quad V \setminus V' \subseteq \bigcup_{v \in V'} N(v). \quad 22$$

23 The minimum dominating set problem consists in determining the minimum cardinality of a 23
24 dominating set of G . Minimum dominating set can be naturally formulated as a minimum 24
25 set cover problem, in which there is a set $N(v) \cup \{v\}$ for each node $v \in V$ (the set of 25
26 nodes dominated by v). Note that the dimension of the minimum set cover formulation of 26
27 minimum dominating set is $k = 2n$, where n is the number of nodes of G (there is one set 27
28 for each node and the set of elements which need to be covered is the set of nodes). 28

29 2. A polynomial-space algorithm 29

30 In this section we describe a $O(1.3803^k)$ polynomial-space recursive algorithm MSC for 30
31 minimum set cover, where k is the dimension of the problem. It immediately follows that 31
32 minimum dominating set can be solved in $O(1.3803^{2n}) = O(1.9053^n)$ time. 32
33 33

34 The algorithm is based on the following simple properties of set covers. 34
35 35

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1   int MSC(S) {
2       if(|S| = 0) return 0;
3       if(∃S, R ∈ S: S ⊂ R) return MSC(S \ {S});
4       if(∃s ∈ U ∃ a unique S ∈ S: s ∈ S) return 1 + MSC(del(S, S));
5       take S ∈ S of maximum cardinality;
6       return min{MSC(S \ {S}), 1 + MSC(del(S, S))};
7   }

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Fig. 1. Recursive algorithm for minimum set cover.

Lemma 1. *Let \mathcal{S} be an instance of minimum set cover. The following properties hold:*

- (1) *If there is a set S in \mathcal{S} which is (properly) included in another set R in \mathcal{S} ($S \subset R$), then there is a minimum set cover which does not contain S . In particular:*

$$msc(\mathcal{S}) = msc(\mathcal{S} \setminus \{S\}).$$

- (2) *If there is an element s of \mathcal{U} which belongs to a unique $S \in \mathcal{S}$, then S belongs to every set cover. In particular:*

$$msc(\mathcal{S}) = 1 + msc(\text{del}(\mathcal{S}, S)).$$

- (3) *For all the remaining $S \in \mathcal{S}$, the following holds:*

$$msc(\mathcal{S}) = \min\{msc(\mathcal{S} \setminus \{S\}), 1 + msc(\text{del}(\mathcal{S}, S))\}.$$

Note that the sets of cardinality one satisfy exactly one of the properties (1) and (2) of Lemma 1.

A basic version of the algorithm is described in Fig. 1. The base case (line 2) is when $|S| = 0$. In that case, $msc(\mathcal{S}) = 0$. Otherwise (lines 3 and 4), the algorithm tries to reduce the dimension of the problem without branching, by applying one of the properties (1) and (2) of Lemma 1. If none of the two properties above applies, the algorithm simply takes (line 5) a set $S \in \mathcal{S}$ of maximum cardinality and branches (line 6) according to property (3) of Lemma 1.

Theorem 1. *Algorithm MSC solves minimum set cover in time $O(1.3803^k)$, where k is the dimension of the problem.*

Proof. The correctness of the algorithm is a straightforward consequence of Lemma 1.

Let $N_h(k)$ denote the number of subproblems of dimension h solved by the algorithm to solve a problem of dimension k . Clearly, $N_h(k) = 0$ for $h > k$ (the subproblems are of dimension lower than the dimension of the original problem). Moreover, $N_k(k) = 1$ (considering the original problem as one of the subproblems). Consider the case $h < k$ (which implies $|S| \neq 0$). If one of the conditions of lines 3 and 4 is satisfied, the algorithm generates a unique subproblem of dimension at most $k - 1$. Thus:

$$N_h(k) \leq N_h(k - 1).$$

1 Otherwise, the algorithm takes a set S of maximum cardinality ($|S| \geq 2$), and it branches on 1
2 the two subproblems $\mathcal{S}_1 = \mathcal{S} \setminus \{S\}$ and $\mathcal{S}_2 = \text{del}(\mathcal{S}, S)$. The dimension of \mathcal{S}_1 is $k - 1$ (one 2
3 set removed from \mathcal{S}). If $|S| \geq 3$, the dimension of \mathcal{S}_2 is at most $k - 4$ (one set removed 3
4 from \mathcal{S} and at least three elements removed from \mathcal{U}). Thus: 4

$$5 \quad N_h(k) \leq N_h(k - 1) + N_h(k - 4). \quad 5$$

7 Otherwise (all the sets in \mathcal{S} have cardinality two), the dimension of \mathcal{S}_2 is $k - 3$ (one set 7
8 removed from \mathcal{S} and two elements removed from \mathcal{U}). Moreover, \mathcal{S}_2 must contain a set S' 8
9 of cardinality one. Then the algorithm has to execute one of the lines 3 and 4 to solve the 9
10 subproblem \mathcal{S}_2 . Thus: 10

$$11 \quad N_h(k) \leq N_h(k - 1) + N_h(k - 3 - 1) = N_h(k - 1) + N_h(k - 4). \quad 11$$

12 A valid upper bound on $N_h(k)$ is $N_h(k) \leq c^{k-h}$, where $c = 1.3802\dots < 1.3803$ is the 12
13 (unique) positive root of the polynomial $(x^4 - x^3 - 1)$. This implies that the total number 13
14 $N(k)$ of subproblems solved is: 14

$$15 \quad N(k) = \sum_{h=0}^k N_h(k) \leq \sum_{h=0}^k c^{k-h} = O(c^k). \quad 15$$

16 The cost of solving a problem of dimension $h \leq k$, excluding the cost of solving the 16
17 corresponding subproblems (if any), is upper bounded by a polynomial $p(k)$ of k . It follows 17
18 that the time complexity of the algorithm is $O(c^k p(k)) = O(1.3803^k)$. \square 18

19 **Corollary 1.** *There is an algorithm which solves minimum dominating set in $O(1.9053^n)$ 19
20 time, where n is the number of nodes in the graph. 20*

21 In line 5 of MSC a set of maximum cardinality is taken. More sophisticated selection 21
22 criteria may lead to improved time bounds. 22

23 3. An exponential-space algorithm 23

24 In this section we show how to reduce the time complexity of MSC via *dynamic pro-* 24
25 *gramming*, at the cost of an exponential space complexity. 25

26 The technique is similar to the one used by Robson in the context of maximum indepen- 26
27 dent set [14]. While solving a problem \mathcal{S} , the same subproblem \mathcal{S}' can appear many times. 27
28 The idea is then to store the solutions of all the subproblems solved in a database. When- 28
29 ever a new subproblem is generated, this database is checked to see whether the solution 29
30 of that subproblem is already available. This way, one ensures that a given subproblem 30
31 is solved at most once. The database can be implemented in such a way that the *query time* 31
32 is logarithmic in the number of solutions stored. 32

33 **Theorem 2.** *Algorithm MSC, modified as above, solves minimum set cover in time 33
34 $O(1.3424^k)$, where k is the dimension of the problem. 34*

1 **Proof.** The correctness of the algorithm follows from Lemma 1. 1

2 Let $N_h(k)$ denote the number of subproblems of dimension $h \in \{0, 1, \dots, k\}$ solved by 2
3 the algorithm to solve a problem of dimension k . From the proof of Theorem 1, $N_h(k) \leq$ 3
4 c^{k-h} , where $c = 1.3802\dots < 1.3803$ is the positive root of the polynomial $(x^4 - x^3 - 1)$. 4
5 Each subproblem \mathcal{S}' is obtained by removing some sets from \mathcal{S} and some elements from \mathcal{U} . 5
6 In other words: 6

$$7 \quad \mathcal{S}' = \text{del}(\mathcal{S} \setminus \mathcal{S}^*, \mathcal{U}^*), \quad 7$$

8 for some $\mathcal{S}^* \subseteq \mathcal{S}$ and $\mathcal{U}^* \subseteq \mathcal{U}$. This implies that $N_h(k) \leq \binom{k}{h}$. Let h' be the largest integer 9
10 in $\{0, 1, \dots, \lfloor k/2 \rfloor\}$ such that $\binom{k}{h'} < c^{k-h'}$. The total number $N(k)$ of subproblems solved 10
11 is: 11

$$12 \quad N(k) = \sum_{h=0}^k N_h(k) \leq \sum_{h=0}^{h'} \binom{k}{h} + \sum_{h=h'+1}^k c^{k-h} = O(c^{k-h'}). \quad 12$$

13 It follows from Stirling's approximation that $N(k)$ is $O(c^{(1-\alpha)k})$, where α satisfies: 13
14
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$$17 \quad c^{1-\alpha} = \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}}. \quad 17$$

18 The cost of each query to the database is polynomial in k . Thus the cost of solving a 18
19 problem of dimension $h \leq k$, excluding the cost of solving the corresponding subproblems 19
20 (if any), is upper bounded by a polynomial $p(k)$ of k . It follows that the time complexity 20
21 of the algorithm is $O(c^{(1-\alpha)k} p(k)) = O(1.3424^k)$. \square 21

22 **Corollary 2.** *There is an algorithm which solves minimum dominating set in $O(1.8021^n)$* 22
23 *time, where n is the number of nodes in the graph.* 23
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