Available online at www.sciencedirect.com
SCIENCE DIRECT®
Electronic Notes in Discrete Mathematics 22 (2005) 157–162
Electronic Notes in DISCRETE MATHEMATICS
www.elsevier.com/locate/endm

On maximum number of minimal dominating sets in graphs

Fedor V. Fomin [1,2]

Department of Informatics
University of Bergen
N-5020 Bergen, Norway

Fabrizio Grandoni [3,4]

Dipartimento di Informatica
Università di Roma "La Sapienza"
Via Salaria 113, 00198 Roma, Italy

Artem V. Pyatkin [5,6]

Department of Informatics
University of Bergen
N-5020 Bergen, Norway

Alexey A. Stepanov [7]

Department of Informatics
University of Bergen
N-5020 Bergen, Norway

1571-0653/$ – see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.endm.2005.06.028

How many subgraphs of a given property can be in a graph on n vertices? This question is one of the basic questions in Graph Theory. For example, the number of 1-factors (perfect matchings) in a simple k-regular bipartite graph on $2n$ vertices is always between $n!(k/n)^n$ and $(k!)^{n/k}$. (The first inequality was known as van der Waerden Conjecture [8] and was proved in 1980 by Egorychev [3] and the second is due to Bregman [1].) Another example is the famous Moon and Moser [7] theorem stating that every graph on n vertices has at most $3^{n/3}$ maximal independent sets. Such combinatorial bounds are of interests not only on their own but also because they are used for algorithm design as well. Lawler [6] used Moon-Moser bound on the number of maximal independent sets to construct $\mathcal{O}((1+\sqrt[3]{3})^n)$ time graph coloring algorithm which was the fastest coloring algorithm for 25 years. Recently Byskov and Eppstein [2] obtain $\mathcal{O}(2.1020^n)$ time coloring algorithm which is also based on a combinatorial bound 1.7724^n on the number of maximal bipartite subgraphs in a graph.

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is called a *dominating set* for G if every vertex of G is either in D, or adjacent to some node in D. A dominating set is *minimal* if all its proper subsets are not dominating. We define $\mathbf{DOM}(G)$ to be the number of minimal dominating sets in a graph G. The *Minimum Dominating Set* problem (`MDS`) asks to find a dominating set of minimum cardinality. Despite of importance of minimum dominating set problem on which hundreds of papers have been written (see e.g. the surveys [4,5] by Haynes et al.), nothing better the trivial $\mathcal{O}(2^n/\sqrt{n})$ bound was known for $\mathbf{DOM}(G)$. In this paper we prove the following

Theorem. *For every graph G on n vertices $\mathbf{DOM}(G) \leq 1.7697^n$.*

Proof. First we reduce `MDS` to the *Minimum Set Cover* problem (`MSC`). In this problem we are given a hypergraph $H = (\mathcal{U}, \mathcal{S})$ with a vertex set $\mathcal{U}$ and an edge set $\mathcal{S}$ of (non-empty) subsets of $\mathcal{U}$. The aim is to determine the minimum

---

[1] Additional support by Norges forskningsråd projects 160778/V30 and 162731/V00
[2] Email: `fomin@ii.uib.no`
[3] Supported by Web-Minds project of the Italian Ministry of University and Research, under the FIRB program
[4] Email: `grandoni@di.uniroma1.it`
[5] Additional support by grants of the Russian Foundation for Basic Research (project code 05-01-00395) and INTAS (project code 04–77–7173)
[6] Email: `Artem.Pyatkin@ii.uib.no`
[7] Email: `ljosha@ljosha.org`

cardinality of a subset $\mathcal{S}^* \subseteq \mathcal{S}$ which *covers* $\mathcal{U}$, i. e. such that

$$\cup_{S\in\mathcal{S}^*} S = \mathcal{U}.$$

A covering is minimal if it contains no smaller covering. We denote by $\mathbf{COV}(H)$ the number of minimal coverings in $H = (\mathcal{U}, \mathcal{S})$.

The problem of finding $\mathbf{DOM}(G)$ can be naturally reduced to finding $\mathbf{COV}(H)$ by imposing $\mathcal{U} = V$ and $\mathcal{S} = \{N[v] \mid v \in V\}$. Note that $N[v] = \{v\} \cup \{u \mid uv \in E\}$ is the set of nodes dominated by v. Thus D is a dominating set of G if and only if $\{N[v] \mid v \in D\}$ is a set cover of $H = (\mathcal{U}, \mathcal{S})$. So, each minimal set cover of H corresponds to a minimal dominating set of G.

Consider now an arbitrary example of the `MSC` problem with a hypergraph $H = (\mathcal{U}, \mathcal{S})$. Denote by s_i the number of edges of cardinality i for $i = 1, 2, 3$ and by s_4 the number of edges of cardinality at least 4 in S. Let $k = |\mathcal{U}| + \sum_{i=1}^{4} \varepsilon_i s_i$ be the *size* of the `MSC` problem $(\mathcal{U}, \mathcal{S})$. Here $\varepsilon_1 = 2.9645, \varepsilon_2 = 3.5218, \varepsilon_3 = 3.9279$, and $\varepsilon_4 = 4.1401$ are the size coefficients for the edges of cardinalities $1, 2, 3$, and at least 4 respectively. Let $\mathbf{COV}(k)$ be the maximum value of $\mathbf{COV}(H)$ among all `MSC` problems of size k. We will prove that $\mathbf{COV}(k) \leq \alpha^k$, where $\alpha \approx 1.11744562 < 1.1175$.

We use induction on k. Clearly, $\mathbf{COV}(0) = 1$. Suppose that $\mathbf{COV}(l) \leq \alpha^l$ for every $l < k$. Let $\mathcal{S}$ be a set of subsets of $\mathcal{U}$ such that the `MSC` problem $(\mathcal{U}, \mathcal{S})$ is of size k. Let $d_2 = \min\{\varepsilon_1, \varepsilon_2 - \varepsilon_1\}, d_3 = \min\{d_2, \varepsilon_3 - \varepsilon_2\}$, and $d_4 = \min\{d_3, \varepsilon_3 - \varepsilon_2\}$. We consider different cases.

Case 0. *There is a vertex* $u \in \mathcal{U}$ *of degree* 1. Since u must be covered by the only set S containing it, we may remove u and S along with all vertices from S and reduce the size of the instance.

Case 1. H *has a vertex* u *belonging to loops (edges of cardinality* 1*) only.* Let $S_1 = S_2 = \ldots = S_r = \{u\}$, where $r \geq 2$ be all the edges containing u. Then every minimal covering should contain exactly one of them. Thus

$$\mathbf{COV}(H) \leq r \cdot \mathbf{COV}(k - r\varepsilon_1 - 1) \leq r\alpha^{k - r\varepsilon_1 - 1}.$$

Case 2. H *contains an edge of cardinality* $r \geq 5$. Let $S = \{u_1, u_2, \ldots, u_r\}$ be such an edge. The number of minimal set covers that do not contain S is at most $\mathbf{COV}(\mathcal{U}, \mathcal{S} \setminus S)$ and the number of minimal set covers containing S is at most $\mathbf{COV}(\mathcal{U} \setminus \{u_1, u_2, \ldots, u_r\}, \mathcal{S}')$. Here $\mathcal{S}'$ consists of all nonempty subsets $S' \setminus \{u_1, u_2, \ldots, u_r\}$ where $S' \in \mathcal{S}$. Therefore

$$\mathbf{COV}(H) \leq \mathbf{COV}(k - \varepsilon_4) + \mathbf{COV}(k - 5 - \varepsilon_4) \leq \alpha^{k-\varepsilon_4} + \alpha^{k-5-\varepsilon_4}.$$

Case 3. H *contains an edge of cardinality* 4. Let $S = \{u_1, u_2, u_3, u_4\}$ be such an edge. Again, $\mathbf{COV}(H) \leq \mathbf{COV}(\mathcal{U}, \mathcal{S} \setminus S) + \mathbf{COV}(\mathcal{U} \setminus \{u_1, u_2, u_3, u_4\}, \mathcal{S}')$.

But now removal of every vertex u_1, u_2, u_3, u_4 from $\mathcal{U}$ reduces the size of the problem by at least $d_4 + 1$, since all edges contain at most 4 vertices and the minimum degree is 2. Thus

$$\mathbf{COV}(H) \le \mathbf{COV}(k - \varepsilon_4) + \mathbf{COV}(k - \varepsilon_4 - 4(d_4 + 1)) \le \alpha^{k-\varepsilon_4} + \alpha^{k-4-\varepsilon_4-4d_4}.$$

We omit the detailed analysis for the next four cases since it is similar to the previous cases but more tedious.

Case 4. *There is $u \in \mathcal{U}$ of degree* 2*.* Then

$$\mathbf{COV}(H) \le \min\{\alpha^{k-1-\varepsilon_1-\varepsilon_2} + \alpha^{k-2-\varepsilon_1-\varepsilon_2-d_3}, 2\alpha^{k-2-2\varepsilon_2}, 2\alpha^{k-2-2\varepsilon_2-d_3} + \alpha^{k-3-2\varepsilon_2-2d_3}\}.$$

Case 5. *H contains an edge of cardinality* 3*.* Then

$$\mathbf{COV}(H) \le \alpha^{k-\varepsilon_3} + \alpha^{k-3-\varepsilon_3-6d_3}.$$

Case 6. *There are $S, S' \in \mathcal{S}$ such that $S' \subset S$.* In this case

$$\mathbf{COV}(H) \le \min\{\alpha^{k-\varepsilon_2} + \alpha^{k-2-\varepsilon_1-\varepsilon_2-3d_2}, \alpha^{k-\varepsilon_2} + \alpha^{k-2-2\varepsilon_2-2d_2}\}$$

Case 7. *There is $u \in \mathcal{U}$ of degree* 3*.* Then

$$\mathbf{COV}(H) \le 3\alpha^{k-2-3\varepsilon_2-2d_2} + 3\alpha^{k-3-6\varepsilon_2} + \alpha^{k-4-6\varepsilon_2}.$$

Case 8. *H does not satisfy any of the conditions from Cases* 1–7*.* In this case H is an ordinary graph of minimum degree at least 4. Let $S = \{u, v\}$ be an edge of H. Denote by $\mathcal{S}_u$ and $\mathcal{S}_v$ the sets of all other edges containing u and v respectively. Clearly, $|\mathcal{S}_u| \ge 3$ and $|\mathcal{S}_v| \ge 3$. By the definition of minimality, if $\mathcal{S}^*$ is a minimal cover containing S then $\mathcal{S}^* \cap \mathcal{S}_u = \emptyset$ or $\mathcal{S}^* \cap \mathcal{S}_v = \emptyset$. So, we have at most $\mathbf{COV}(k - \varepsilon_2)$ minimal covers that do not contain S and at most $2 \cdot \mathbf{COV}(k - 2 - 4\varepsilon_2 - 3d_2)$ covers containing S. Then

$$\mathbf{COV}(H) \le \alpha^{k-\varepsilon_2} + 2\alpha^{k-2-4\varepsilon_2-3d_2}.$$

Summarizing, we have the following inequalities:

$$\alpha^k \geq \max \begin{cases} r\alpha^{k-r\varepsilon_1-1}, \ r \geq 2 \\ \alpha^{k-\varepsilon_4} + \alpha^{k-5-\varepsilon_4} \\ \alpha^{k-\varepsilon_4} + \alpha^{k-4-\varepsilon_4-4d_4} \\ \alpha^{k-1-\varepsilon_1-\varepsilon_2} + \alpha^{k-2-\varepsilon_1-\varepsilon_2-d_3} \\ 2\alpha^{k-2-2\varepsilon_2} \\ 2\alpha^{k-2-2\varepsilon_2-d_3} + \alpha^{k-3-2\varepsilon_2-2d_3} \\ \alpha^{k-\varepsilon_3} + \alpha^{k-3-\varepsilon_3-6d_3} \\ \alpha^{k-\varepsilon_2} + \alpha^{k-2-\varepsilon_1-\varepsilon_2-3d_2} \\ \alpha^{k-\varepsilon_2} + \alpha^{k-2-2\varepsilon_2-2d_2} \\ 3\alpha^{k-2-3\varepsilon_2-2d_2} + 3\alpha^{k-3-6\varepsilon_2} + \alpha^{k-4-6\varepsilon_2} \\ \alpha^{k-\varepsilon_2} + 2\alpha^{k-2-4\varepsilon_2-3d_2} \end{cases}$$

It could be checked (by using computer) that all of them hold for given α and $\varepsilon_i, \ i = 1, 2, 3, 4$. Therefore, $\mathbf{COV}(k) \leq \alpha^k < 1.1175^k$. But for any graph G on n vertices the corresponding instance of `MSC` problem has size at most $|\mathcal{U}| + \varepsilon_4|\mathcal{S}| = (1+\varepsilon_4)n$. Thus $\mathbf{DOM}(G) \leq \mathbf{COV}((1+\varepsilon_4)n) < 1.1175^{5.1401n} \leq 1.7697^n$, finishing the proof. □

Note finally that the best known lower bound for $\mathbf{DOM}(G)$ is $15^{n/6} \approx 1.5704^n$ (consider $n/6$ disjoint copies of the octahedron). This example was found by Dieter Kratsch.

Acknowledgements. We thank Dieter Kratsch for attracting our attention to this problem.

References

[1] L. M. Brègman. Certain properties of nonnegative matrices and their permanents. *Doklady Akademii Nauk BSSR*, 211:27–30, 1973.

[2] Makholm Byskov and David Eppstein. An algorithm for enumerating maximal bipartite subgraphs. *manuscript*, 2004.

[3] G. P. Egorychev. Proof of the van der Waerden conjecture for permanents. *Sibirsk. Mat. Zh.*, 22(6):65–71, 225, 1981.

[4] Teresa W. Haynes and Stephen T. Hedetniemi, editors. *Domination in graphs.* Marcel Dekker Inc., New York, 1998.

[5] Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater. *Fundamentals of domination in graphs.* Marcel Dekker Inc., New York, 1998.

[6] E. L. Lawler. A note on the complexity of the chromatic number problem. *Information Processing Lett.*, 5(3):66–67, 1976.

[7] J. W. Moon and L. Moser. On cliques in graphs. *Israel Journal of Mathematics*, 3:23–28, 1965.

[8] B. van der Waerden. Problem 45. *Jahresber. Deutsch. Math.-Verein.*, 35:117, 1926.