Contents

9.3 Synopsis of discrete spatial interaction models

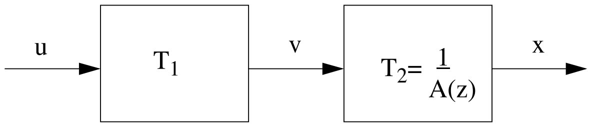

Figure 9.1: Synthesis of the random field.

We restrict ourselves to discrete spatial interaction models with real or integer valued grey level intensities. The random field $\boldsymbol{X}$ is *generated* by transformations of a random sequence $\boldsymbol{u}$ as samples of a certain stochastic process $\boldsymbol{U}$, which will be specified later. In Fig. 9.1 is presented the diagram for the synthesis of the random field $\boldsymbol{X}$. The stochastic model of $\boldsymbol{X}$ is defined by the transformations T_1 and T_2 and the characteristics of $\boldsymbol{U}$. T_1 and T_2 are not limited by any physical realizability considerations as is the case in system theory. Transformation T_2 is defined as

$$T_2 = \frac{1}{A(z)} \tag{9.1}$$

where

$$A(z) = 1 - \sum_{r \in \mathcal{N}_s} \theta_r \, z^r \tag{9.2}$$

θ_r are the model parameters, and the following notation is used

$$z^r \, x_s = x_{s+r} = x_{s_1+r_1, s_2+r_2} \tag{9.3}$$

$r = (r_1, r_2)$ and $s = (s_1, s_2)$ are sites of the lattice $\mathcal{L}$, and $\mathcal{N}_r$ is the vicinity of site r. We will also use the alternative definition for the spectral representation

$$A_D(r) = A\left(z = e^{j\,\frac{2\pi\, r}{M}}\right) \tag{9.4}$$

where $M \times M$ is the size of the lattice $\mathcal{L}$.

The transformation T_1 has different expressions for different classes of models. In Table 9.1 we give the definition for only three such models.

9.3.1 Analysis of Markov models

In this lecture we restrict the discussion to only a particular class of Markov models. The transformation T_2 will be written as:

$$x_s - \sum_{r \in \mathcal{N}_s} \theta_r \, x_{s+r} \; = \; \sqrt{\sigma^2}\, v_s \tag{9.5}$$

Model	T_1
Simultaneous autoregressive (SAR)	$T_1 = 1$
SAR moving average (SARMA)	$T_1 = B(z) = 1 + \sum_{r \in \mathcal{N}_s} \beta_r z^r$
Conditional Markov (MRF)	T_1 is defined implicitly from the power spectrum $$\Phi_v(\omega) = A(e^{jw})$$ The following notation is used $$T_1 = \sqrt{A(z)}$$ **Obs.:** If there is an $A_1(z)$ such that $$A_1(z)\, A_1(z) \; = \; A(z)$$ the model is a SAR model. If $A(z)$ has no factorization, the definition has no meaning. In one dimension a symmetric polynomial $A(z)$ is always factorizable. This is one of the key differences and difficulties of the two-dimensional problems.

Table 9.1: Stochastic models and the associated transformations.

with $\theta_r = \theta_{-r}$, the model is defined on a symmetric neighborhood. The process $\boldsymbol{v}$ and the transformation T_1 are implicitly defined by the following conditions:

$$\mathcal{E}[v_s|x_r;\, r \neq s] = 0 \tag{9.6}$$

i.e. v_s is independent of x_r for sites $r \neq s$. Hence

$$\mathcal{E}[x_s|x_r;\, r \neq s] = \sum_{r \in \mathcal{N}_s} \theta_r\, x_{s+r} \tag{9.7}$$

$\boldsymbol{X}$ is a MRF with respect to the symmetric neighbor set $\mathcal{N}$ on the lattice $\mathcal{L}$.

If the process v is Gaussian we can write:

$$\begin{aligned} p(x_s|x_r;\, r \neq s) &= p(x_s|x_{s+r};\, r \in \mathcal{N}_s) \\ &= \frac{1}{\sqrt{2\pi\,\sigma^2}} e^{-\frac{\left(x_s - \sum_{r \in \mathcal{N}_s} \theta_r\, x_{s+r}\right)^2}{2\sigma^2}} \end{aligned} \tag{9.8}$$

Further we assume X to be stationary and evaluate its power spectral density. We will use the Wiener-Khintchine theorem (Eq. 7.31). Eq. 9.5 is multiplied by $x_{s+s'}$ and the expectation of the product is evaluated. Using the statistical independence in Eq. 9.6 we obtain the autocorrelation function

$$R_X(r') = \sum_{r \in \mathcal{N}} \theta_r\, R_X(r + r') \,+\, \sigma^2 \delta(r') \tag{9.9}$$

where

$$\delta(r') = \begin{cases} 0 & \text{if} \quad r' \neq (0,0) \\ 1 & \text{if} \quad r' = (0,0) \end{cases} \tag{9.10}$$

After evaluation of the Fourier transform of both sides, we obtain:

$$\left(1 - \sum_{r \in \mathcal{N}} \theta_r\, z_r\right) \Phi_{\boldsymbol{X}}(\omega) = \sigma^2 \tag{9.11}$$

Thus

$$\Phi_{\boldsymbol{X}}(\omega) = \frac{\sigma^2}{1 - \sum_{r \in \mathcal{N}} \theta_r\, \exp\left(j\left(\frac{2\pi}{M} r_1\, \omega_1 + \frac{2\pi}{M} r_2\, \omega_2\right)\right)} \tag{9.12}$$

where $\omega = (\omega_1, \omega_2)$, with $0 \leq \omega_1, \omega_2 \leq M - 1$. Using the model definition we observe that

$$\Phi_{\boldsymbol{X}}(\omega) = \frac{\sigma^2}{||A \exp(j\omega)||^2} \Phi_{\boldsymbol{v}}(\omega) \tag{9.13}$$

which defines the process $\boldsymbol{v}$.

9.3.2 A matrix formalism

Until now we wrote all equations for a generic site s of the lattice $\mathcal{L}$. The characterization of the random field, as given in Eq. 9.5, is actually written for all $M \times M$ sites of the lattice $\mathcal{L}$. Further we assume the lattice $\mathcal{L}$ to have circular symmetry as introduced in section 8.7, and use the stack vector representation $\boldsymbol{x}$ of the image $x_s, s \in \mathcal{L}$. Thus we can write the model Eq. 9.5 in matrix form:

$$\sqrt{\mathcal{A}(\theta)}\boldsymbol{x} = \sigma\, \boldsymbol{v} \tag{9.14}$$

Where $\boldsymbol{v}$ is the stack vector used for the representation of the process $\boldsymbol{v}$, and $\mathcal{A}(\theta)$ is a symmetric block circulant matrix.

$$\mathcal{A}(\theta) = \begin{bmatrix} \mathcal{A}_{1,1} & \mathcal{A}_{1,2} & \cdots & \mathcal{A}_{1,M} \\ \mathcal{A}_{1,M} & \mathcal{A}_{1,1} & \cdots & \mathcal{A}_{1,M-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{1,2} & \mathcal{A}_{1,3} & \cdots & \mathcal{A}_{1,1} \end{bmatrix} \tag{9.15}$$

with each $\mathcal{A}_{1,i}$ a $M \times M$ circulant matrix,

$$\mathcal{A}_{1,i} = \mathcal{A}^T_{1,M-i+2} \tag{9.16}$$

Now we can write the expression for the joint p.d.f. of the entire image assuming the knowledge of the p.d.f. of the process $\boldsymbol{v}$.

$$p(\boldsymbol{x}) = \frac{1}{\sigma^{\frac{M^2}{2}}\, |\det \mathcal{A}(\theta)|^{\frac{1}{2}}} \prod_{r\in\mathcal{L}} p(\, v_r\,) \tag{9.17}$$

where $\det \mathcal{A} = \prod_{r\in\mathcal{L}} ||A_D(r)||^2$. If $\boldsymbol{v}$ is a Gaussian process

$$p(\boldsymbol{x}) = \frac{\exp\left(-\frac{1}{2\sigma^2}\, x^T \mathcal{A}(\theta)\, x\right)}{(2\pi\sigma^2)^{\frac{M^2}{2}} \prod_{r\in\mathcal{L}} A_D^{\frac{1}{2}}(r)} \tag{9.18}$$

9.3.3 Gauss Markov Random Field

Thus, in summary, a Gauss Markov Random Field (GMRF) defined on the lattice $\mathcal{L}$ and having symmetric neighborhood is characterized by

$$x_s \;=\; \sum_{r\in\mathcal{N}_s} \theta_r\, x_{s+r} + \sqrt{\sigma^2}\, v_s \tag{9.19}$$

$$\sqrt{\mathcal{A}(\theta)}\boldsymbol{x} \;=\; \sigma\, \boldsymbol{v} \tag{9.20}$$

With $\boldsymbol{v}$ a zero mean, unit variance Gaussian noise having

$$\mathcal{E}[v_r \cdot v_{r+s}] = \begin{cases} 1 & \text{if} \quad r = (0,0) \\ -\theta_r & \text{if} \quad r \in \mathcal{N}_s \\ 0 & else \end{cases} \tag{9.21}$$

and $\theta_r = \theta_{-r}$. The covariance matrix C_X of X is positive definite. The joint p.d.f. of X is

$$p(\boldsymbol{x}) = \frac{1}{\sigma^{\frac{M^2}{2}} |\det C_X|^{\frac{1}{2}}} e^{-\frac{1}{2\sigma^2}\boldsymbol{x}^T C_X^{-1}\boldsymbol{x}} \tag{9.22}$$

The Markov property

$$\begin{aligned} p(x_s | x_r;\, r \neq s) &= p(x_s | x_{s+r};\, r \in \mathcal{N}_s) \\ &= \frac{1}{\sqrt{2\pi\,\sigma^2}} e^{-\frac{\left(x_s - \sum_{r\in\mathcal{N}_s} \theta_r\, x_{s+r}\right)^2}{2\sigma^2}} \end{aligned} \tag{9.23}$$

and the power spectrum is

$$\Phi_{\boldsymbol{X}}(\omega) = \frac{\sigma^2}{1 - \sum_{r\in\mathcal{N}} \theta_r\, \exp\left(j\left(\frac{2\pi}{M}\, r_1\omega_1 + \frac{2\pi}{M}\, r_2\omega_2\right)\right)} \tag{9.24}$$

A GMRF is a special case of GRF. A Gaussian process is a special case of GMRF whose Gibbs energy consists only of single site clique potentials. It has no contextual interaction.

Bibliography

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