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Inversion of ground-based radiometric data by Kalman filtering

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Retrieval of atmospheric vertical temperature profiles from ground-based radiometric observations requires shrewdness and judicious choice of parameters to surmount the noxious effects of the ill-posed nature of the inversion. Kalman linear estimation, already successfully used in satellite microwave sounding of atmospheric temperature, can be also applied to infer the thermal state of the lower troposphere from ground-based infrared measurements. After a short survey of the relevant formalism the implementation of the Kalman filter is discussed with regard to the transition operator and to the error covariances. In particular, the measurement errors are considered in some detail, and their dependence on the atmospheric dynamics is pointed out. The attainable spatial resolution is compared with that of another commonly used inversion technique, and, finally, a set of temperature profiles estimated by the Kalman algorithm from a sequence of successive radiometric measurements is reported.

1. INTRODUCTION

When estimating atmospheric temperature profiles, special care is required in the inversion of radiometric data in order to minimize the inherent instability of the solution. The situation is worse when the pertinent radiative transfer kernels are particularly smooth, as can be the case for ground-based radiometry. A number of methods have been devised, by which erratic solutions of ill-posed problems can be curbed within reasonably significant patterns. Commonly used techniques include the regularization of the sought solution [*Tihonov,* 1963; *Twomey,* 1963], statistical methods [*Strand and Westwater,* 1968; *Rodgers,* 1975], synthetic averaging kernels [*Backus and Gilbert,* 1968; 1970], relaxation procedures [*Chahine,* 1970], and 'shaped' iteration processes [*Strand,* 1974]. To be successful, each technique has its own requirements. Reasonable regularization can be achieved by judiciously choosing the smoothing parameters, while the availability of climatological data is demanded by statistical methods. A knowledge of the statistics of the measurement noise can also be a major requirement, as is the case for Backus-Gilbert inversion. The number of steps, in connection with initial guesses of profiles as well as with the chosen shaping operators, can affect the significance of the solution obtained through relaxation and iterative procedures.

Recently, Kalman filtering has been tested as a novel technique for retrieving atmospheric temperature profiles from satellite microwave spectrometric observations [*Ledsham and Staelin,* 1978]. In its recursive scheme, both deterministic information and statistical elements are blended to produce an optimal estimate of the unknown quantities from the available measurements [*Sorenson and Stubberud,* 1970]. The iterative frame of the filter allows one to propagate the estimate from one time to the other (or, equivalently, from one place to an adjacent one) by using the information which is available on the deterministic state transition in space or in time of the process under observation. Both statistics of random deviations of the actual values from the predicted values and the known measurement noise come then into the algorithm to settle the optimum balance between the prediction and the innovation brought by the measurements. Although occasional poor performance could occur at the start of its recursive implementation, Kalman filtering was noticed to achieve overall significant accuracy in atmospheric temperature retrieval from satellite brightness measurements.

In this paper, the use of the Kalman filter to estimate vertical profiles of low-altitude tropospheric temperature from ground-based infrared radiometric measurements is considered. After a short summary of the relevant algebra the filter implementation is discussed in some detail with regard to the structure of the transition operator

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and to the construction of the error covariances of the a priori expectations and of the propagated estimates. The statistical properties of the measurement errors are then examined, and their dependence on the atmospheric dynamics is investigated. The attainable spatial resolution is compared with that of another commonly used inversion technique, and, finally, a set of temperature profiles is estimated from a sequence of radiometric measurements taken at successive times on a 24-hr period.

2. THERMAL PROFILES AND KALMAN FILTERING

The use of Kalman-Bucy equations in the inference of atmospheric temperature profiles has been already discussed elsewhere [*Ledsham and Staelin*, 1978]. In the following, only a short summary of the relevant algebra will be given. The reader interested in the underlying mathematical theory, as well as in more details about the temperature retrieval problem, is referred to the aforecited work and bibliography quoted therein, as well as to papers on ground-based radiometry [*Wang et al.*, 1975; *Westwater et al.*, 1975].

When a narrow field-of-view ground-based radiometer points toward the atmosphere at elevation angle ϑ, its output voltage is given by

$$V_r(\vartheta) = K_r \int_0^L W_0(\vartheta, r) B_0(r)\, dr \qquad (1)$$

where K_r is an instrument factor, and both the radiative transfer kernel W_0 and the Planck function B_0 are considered at a reference wavelength λ_0 within the spectral band of the instrument. The atmosphere is considered to extend up to a maximum distance L and is assumed to be nonscattering and in local thermodynamic equilibrium. The irregular motion of the air produces fluctuations of both temperature $T(r)$ and absorption coefficient $\alpha(r)$ at distance r from the instrument, and hence, in turn, W_0 and B_0 are space-time random functions. When they are split into average components and zero mean fluctuating parts,

$$\begin{aligned} W_0(\vartheta, r, t) &= \bar{W}_0(\vartheta, r) + W'(\vartheta, r, t) \\ B_0(r, t) &= \bar{B}_0(r) + B'(r, t) \end{aligned} \qquad (2)$$

the output of the radiometer is written as

$$V_r(\vartheta, t) = K_r \int_0^L \bar{W}_0(\vartheta, r) \bar{B}_0(r)\, dr + V_r'(\vartheta, t) \qquad (3)$$

where, when small fluctuations about averages are considered, the random part of the output is

$$V_r'(\vartheta, t) \cong K_r \int_0^L [\bar{W}_0(\vartheta, r) B'(r, t) + W'(\vartheta, r, t) \bar{B}_0(r)]\, dr + V_n'(t) \qquad (4)$$

and results from both atmospheric fluctuations and instrumental noise V_n'. The average part of the radiometric output, which if the integrand is considered as a function of height z above ground rather than of distance r and if the atmosphere extends up to a maximum height H, can be written as

$$\bar{V}_r(\vartheta) \cong K_r \int_0^H \bar{W}_0(\vartheta, z) \bar{B}_0[\bar{T}(z)]\, dz \qquad (5)$$

contains the useful information on the temperature profile, since only the average radiative transfer kernel is a priori known, and, moreover, the definition itself of (noninstantaneous) vertical profile implies that the short-term fluctuations of temperature be smoothed out. To estimate the thermal profile, integral equation (5) must be solved with respect to the Planck function $\bar{B}_0(z)$ for a known (measured) $\bar{V}_r(\vartheta)$. However, in general, feasible measurement procedures do not allow the average output to be exactly known because of the effect of the random component (4), which unavoidably limits the accuracy of measurements. As a consequence, an estimate of $\bar{B}_0(z)$ must be attempted by solving the Fredholm integral equation of the first kind (5) in which the known function is corrupted by some measurement noise. This kind of ill-posed problems does not always possess a stable solution, in the sense that depending on the form of the kernel, the errors in the known data can produce substantial perturbation in the solution up to the point that this latter eventually loses any meaning. This is, in fact, the case for (5), since ground-based radiative transfer kernels are fundamentally ill conditioned.

To overcome the difficulties inherent in the integral equation approach to the inversion of noisy data, the problem may be considered from a different, although essentially equivalent, point of view. The atmosphere is regarded as a linear, stochastic dynamical system, and the measurements which carry the information about its thermal state are assumed to be linearly related to the thermal state vector itself:

$$\mathbf{WB} = \mathbf{V} + \mathbf{N} \tag{6}$$

In (6), $\mathbf{V}$ is the vector of measurements, $\mathbf{B}$ is the state vector, $\mathbf{N}$ represents the noise in the measurements, and $\mathbf{W}$ is the observation matrix. Equation (6) is readily recognized to be the matrix equation to which integral equation (5) reduces when the practically relevant discrete case is considered and the noise is explicitly included. This observation leads one to identify $\mathbf{W}$ as the matrix of the discrete weighting functions derived from the corresponding radiative transfer kernels, $\mathbf{B}$ as the vector of the Planck functions for the various heights above the ground, $\mathbf{V}$ as the vector of the radiometer output measured at different elevation angles, and $\mathbf{N}$ as the corresponding noise term. Kalman linear estimation procedures can be followed to infer an unbiased, minimum variance estimate of $\mathbf{B}$, given data $\mathbf{V}$:

$$\hat{\mathbf{B}} = \hat{\mathbf{B}}_p + \mathbf{K}(\mathbf{V} - \hat{\mathbf{V}}_p) \tag{7}$$

where subscript p denotes the a priori expectation and the caret, in general, denotes the estimate. Kalman gain $\mathbf{K}$ has the following form:

$$\mathbf{K} = \mathbf{P}_p\mathbf{H}^T(\mathbf{H}\mathbf{P}_p\mathbf{H}^T + \mathbf{R})^{-1} \tag{8}$$

where $\mathbf{P}_p$ is the error covariance of the a priori expected profile $\hat{\mathbf{B}}_p$ and $\mathbf{R}$ is the error covariance matrix of the measurement vector $\mathbf{V}$. It can be remarked that Kalman gain settles an optimum balance between a priori estimate and measurement innovation according to the precision of the former and the accuracy of the latter. Equations (7) and (8) themselves allow the thermal profile to be inferred from inversion of radiometric data. However, the linear stochastic system approach contains additional resources, since it enables estimates to be propagated in time, as well as in space. The evolution of the atmospheric thermal state is assumed to be described by

$$\mathbf{B}_{i+1} = \mathbf{\Phi}_{i,i+1}\mathbf{B}_i + \mathbf{T}_i \tag{9}$$

where $\mathbf{\Phi}_{i,i+1}$ is the matrix which expresses the deterministic transition between successive states, here denoted by subscripts i and $i + 1$, and $\mathbf{T}_i$ is the random component of the transformation, characterized by the covariance matrix $\mathbf{Q}_i$. Covariance $\mathbf{P}_{i+1}$ of the error affecting the state vector $\mathbf{B}_{i+1}$ can also be related to the previously calculated covariance $\mathbf{P}_i$ through the following equation:

$$\mathbf{P}_{i+1} = \mathbf{\Phi}_{i,i+1}\mathbf{P}_i\mathbf{\Phi}^T_{i,i+1} + \mathbf{Q}_i \tag{10}$$

It is spontaneous to identify the state vector and the corresponding covariance propagated by (9) and (10) with a priori expectation $\hat{\mathbf{B}}_p$ and covariance $\mathbf{P}_p$ so that data eventually available about any new state can be processed through (7) and (8), taking full advantage of the information supplied by the past measurements. A recursive process is thus triggered, through which successive thermal profiles are estimated from a sequence of radiometric measurements, with adequate exploitation of the known past history of the atmospheric thermal state.

3. DESIGN OF THE FILTER

Implementing the algorithm requires that three covariance matrices and the deterministic transition matrix be specified. The following discussion is centered mainly on the solutions that have been adopted for a ground-based infrared experiment conducted by the authors in the Mediterranean area. Obviously, different experimental conditions and measuring procedure may require variations in the determination of the filter parameters.

3.1. *A priori expectation and errors.* To initiate the filter, an a priori expected thermal profile is necessary. When climatological statistics are not available for the site of the experiment, model atmospheres [*Valley,* 1965] may be used. Since this experiment was carried out in Rome, Italy, synoptic radio soundings taken at the Leonardo da Vinci Airport in Fiumicino at about 20 km from the site of the experiment appear to yield more accurate a priori information than the standard atmospheric models. Eighty-three temperature profiles measured by radiosonde in the same season as that of the experiment were averaged over 4 yr to form the expected initial thermal profile. Since the effect of the urban area on the local meteorological variables can be noticeable [*Colacino and Dell'Osso,* 1978], the lowest level temperatures were corrected by taking into account the temperature values measured on a baroque tower in the center of the city of Rome. The error covariance matrix of this a priori estimate was determined in a straightforward way by comparing the temperatures of each individual profile with the means calculated at the various levels.

3.2. *Transition matrix and errors.* The transition matrix must be constructed so as to linearly relate the temperature profile at a given time to a previous one. If temperature statistics for each pair of times are available, solution of linear equation systems yields rows (or columns) of the transition matrix. However, in the most common case, only profiles measured 12 hr apart are available. On the other hand, the transition matrix represents the deterministic link between profiles at different times, and hence models that suitably describe the temporal evolution of the thermal state of the atmosphere can be used alternatively. To form the transition matrix to be used in this experiment, a simple linear diffusion model [*Laikhtman,* 1964] with height-dependent diffusion coefficient has been worked out, which yields the temperature profile as a function of time for assigned boundary (at ground and at infinite height) conditions. Pairs of profiles 3 hr apart formed the data basis for the construction of the transition matrix. The accuracy of the a priori information obtainable by this procedure clearly depends on how realistic the adopted model is. Eventual additional information, such as that obtainable by active remote sensing techniques [*Westwater,* 1978], can be incorporated in more complex models to forecast the deterministic evolution of relevant features of the profile, such as elevated (subsidence) inversions.

The errors affecting the propagated estimates (plant noise) can be determined by a comparison between estimates and actually measured profiles or, when feasible, by evaluating the overall accuracy of the used model.

3.3. *Measurement errors.* As has been shown in section 2, the accuracy of the measurements is limited by both the atmospheric fluctuations and the instrumental noise. A generic element of the measurement error covariance matrix is given by

$$R_{ij} = \langle V'(\vartheta_i, t)\, V'(\vartheta_j, t + \tau_{ij})\rangle \tag{11}$$

if the measurement at elevation angle ϑ_i is performed at time t and the measurement at angle ϑ_j refers to time $t + \tau_{ij}$; angle brackets denote average. According to (4), (11) becomes

$$R_{ij} = \Big\langle \Big\{ K_r \int_0^{L_i} [\bar{W}_0(\vartheta_i, r) B'(r, t) + W'(\vartheta_i, r, t)\bar{B}_0(r)]\, dr + V_n'(t) \Big\} \cdot \Big\{ K_r \int_0^{L_j} [\bar{W}_0(\vartheta_j, r) B'(r, t + \tau_{ij}) + W'(\vartheta_j, r, t + \tau_{ij})\bar{B}_0(r)]\, dr + V_n'(t + \tau_{ij}) \Big\} \Big\rangle \tag{12}$$

Instrumental noise V_n', which takes into account the noise of the detector and of the electronic system, is uncorrelated with the atmospheric fluctuations. Local fluctuations of the Planck function, in turn, are weakly correlated with the fluctuations of the weighting function, which result from the effect of the irregular variations of the local transmittance piled up along the radiometer beam. Therefore it seems reasonable to assume R_{ij} to be formed by three terms only:

$$\begin{aligned} R_{ij} \cong\; & K_r^2 \int_0^{L_i}\int_0^{L_j} \bar{W}_0(\vartheta_i, r')\bar{W}_0(\vartheta_j, r'') \cdot \langle B'(r', t) B'(r'', t + \tau_{ij})\rangle\, dr'\, dr'' \\ & + K_r^2 \int_0^{L_i}\int_0^{L_j} \langle W'(\vartheta_i, r', t) W'(\vartheta_j, r'', t + \tau_{ij})\rangle \cdot \bar{B}_0(r')\bar{B}_0(r'')\, dr'\, dr'' + \langle V_n'(t) V_n'(t + \tau_{ij})\rangle \end{aligned} \tag{13}$$

The first term derives from the fluctuations of the Planck function, that is, from the irregular variations of the atmospheric temperature. When the fluctuations are small with respect to the average quantities, the space-time correlation function of the Planck function can be linearly related to the space-time correlation function of the atmospheric temperature fluctuations. For the elements on the main diagonal of the covariance matrix, $\tau_{ij} = 0$, so that by a procedure analogous to that followed elsewhere [*Basili and Solimini,* 1978; *Ciotti et al.,* 1979] the first term of R_{ii} can be related to the variance σ_T^2 and to the spatial spectral density of temperature $\Phi_T(\chi)$:

$$\begin{aligned} & K_r^2 \int_0^{L_i}\int_0^{L_i} \bar{W}_0(\vartheta_i, r')\bar{W}_0(\vartheta_i, r'') \cdot \langle B'(r', t) B'(r'', t)\rangle\, dr'\, dr'' \\ & \cong \frac{K_r^2\lambda_0^8}{\pi k^2 c^2} \Big\{ \int_0^{L_i/2} [\bar{B}_0(R)/\bar{T}(R)]^4 \cdot \exp[2hc/\lambda_0 k\bar{T}(R)]\, g_{W_i}(R)\sigma_T^2(R) \cdot R\Big[\int_{-\pi/4R}^{\pi/4R} \Phi_T(\chi)\, d\chi\Big] dR \\ & \quad + \int_{L_i/2}^{L_i} [\bar{B}_0(R)/\bar{T}(R)]^4 \cdot \exp[2hc/\lambda_0 k\bar{T}(R)]\, g_{W_i}(R)\sigma_T^2(R) \cdot (L_i - R)\Big[\int_{-\pi/4(L_i-R)}^{\pi/4(L_i-R)} \Phi_T(\chi)\, d\chi\Big] dR \Big\} \cdots \end{aligned} \tag{14}$$

In (14), k is Boltzmann's constant, c is the velocity of light, h is Planck's constant, and function g_{W_i}, defined as

$$g_{W_i} = \bar{W}_0(\vartheta_i, r')\,\bar{W}_0(\vartheta_i, r'') \tag{15}$$

has been assumed to depend on R rather than on r' and r'' separately. This assumption approaches reality for atmospheric parameters varying nearly exponentially and for heights above the ground which are small with respect to the characteristic heights of the exponential model. An isotropic, locally homogeneous temperature field with smoothly varying mean characteristics has also been assumed. Equation (14) points out that the contribution of the temperature fluctuations to the error covariance of the radiometric measurements results from the local temperature variance integrated along the radiometer beam with weights which depend both on the atmospheric absorption and on the local average temperature. In addition, for a given variance the shape of the spatial spectral density of thermal irregularities also affects the error covariance. It should be noted that the contribution of thermal fluctuations occurring at $R > L_i/2$ is generally exiguous, so that the second double integral in (14) is usually negligible with respect to the first one. For elements outside the main diagonal of the matrix (for which $\tau_{ij} \neq 0$) the temperature correlation function can still be expanded in terms of its spatial spectral density by regarding the time delay τ_{ij} as a parameter [*Panchev*, 1971]. Formulas substantially analogous to (14), although algebraically cumbersome, can be obtained under similar hypotheses. Since the time-dependent spectral density $\hat{\Phi}_T(\chi, \tau_{ij})$ generally decreases with increasing τ_{ij}, thermal fluctuations give their maximum contribution to elements on the main diagonal of the error covariance matrix, whereas their effect on the off-diagonal elements diminishes toward the corners of the matrix.

The second term of (13), which takes into account the variations of the weighting functions, depends on the average atmospheric temperature. For a given average absorption, a nearly isothermal situation generally minimizes the effect of the weight fluctuations. Therefore it is expected that the matrix elements that correspond to the lowest elevation angles receive the least contribution from these fluctuations.

Finally, the properties of the third term can be straightforwardly appreciated from the characteristics of the noise of the detector and of the processing electronics.

4. SPATIAL RESOLUTION

A major feature of an inversion technique is its spatial resolution, i.e., its capability of discriminating between different values taken by the observed variable at different heights. The resolving power is ultimately determined by the weighting functions pertinent to the measurement. Meteorological data obtained from averages of Fiumicino seasonal radio soundings (see section 3.1) were inserted into the Lowtran 3B code [*Selby et al.*, 1976] to calculate the weighting functions for the infrared bands used in this experiment. Figure 1 shows the weights relative to a 10% wide band centered about wavelength $\lambda_0 = 13.75$ μm for various elevation angles and as a function of height.

To estimate the resolution of the inversion procedure, the radiance produced at ground level by a δ function temperature distribution has been computed for different elevation angles and various heights of the temperature impulse. Such a synthetic data set was then inverted by the Kalman algorithm, thus obtaining the shape of the scanning functions (or averaging kernels) at the different atmospheric levels. Figure 2 reports the scanning functions at ten different levels as produced by the Kalman inversion of 18 synthetic radiances. For comparison, the Backus-Gilbert averaging kernels for the same set of weights is also reported in the figure. Note that a small (0.1%) measurement error had to be assumed in order to prevent instability of the numerical solutions.

5. AN EXAMPLE

Atmospheric radiance was measured by an infrared ground-based radiometer located on the electrical engineering building near the center of Rome, Italy. The elevation angle was scanned from 1° above the horizon to 36° for about 10 min every 3 hr over a 24-hr period, which started at 12.00 GMT (1300 LT) on July 12, 1977 and ended at noon (GMT) of the following day. Measurements in the 13.75-μm band were inverted by the Kalman routine to infer the vertical temperature profile of the assumed horizontally layered lower troposphere. As already stated in section 3.1, the average of temperature profiles measured in summer at 12.00 GMT at Fiumicino Airport and suitably corrected

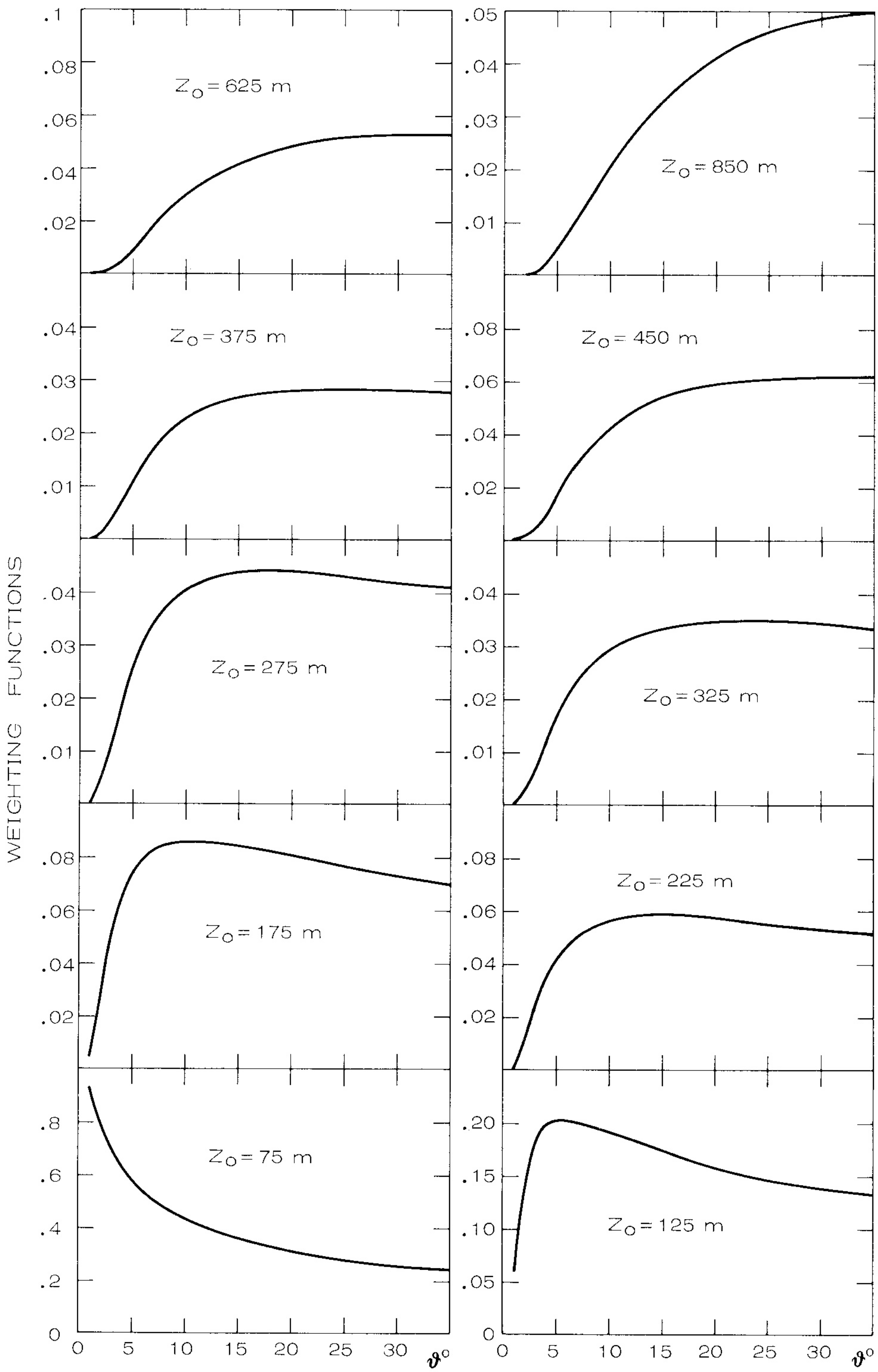

Fig. 1. Average summer weighting functions for atmospheric layers at various elevations z_0 versus elevation angle ϑ. Thickness of layers is $\Delta z = 50$ m for heights up to $z_0 = 375$ m; then $\Delta z = 150$, 200, and 350 m for $z_0 = 450$ m, 625 m, and 850 m, respectively. Center wavelength $\lambda_0 = 13.75$ μm, 10% bandwidth. Note that the scale changes as z_0 and/or Δz vary.

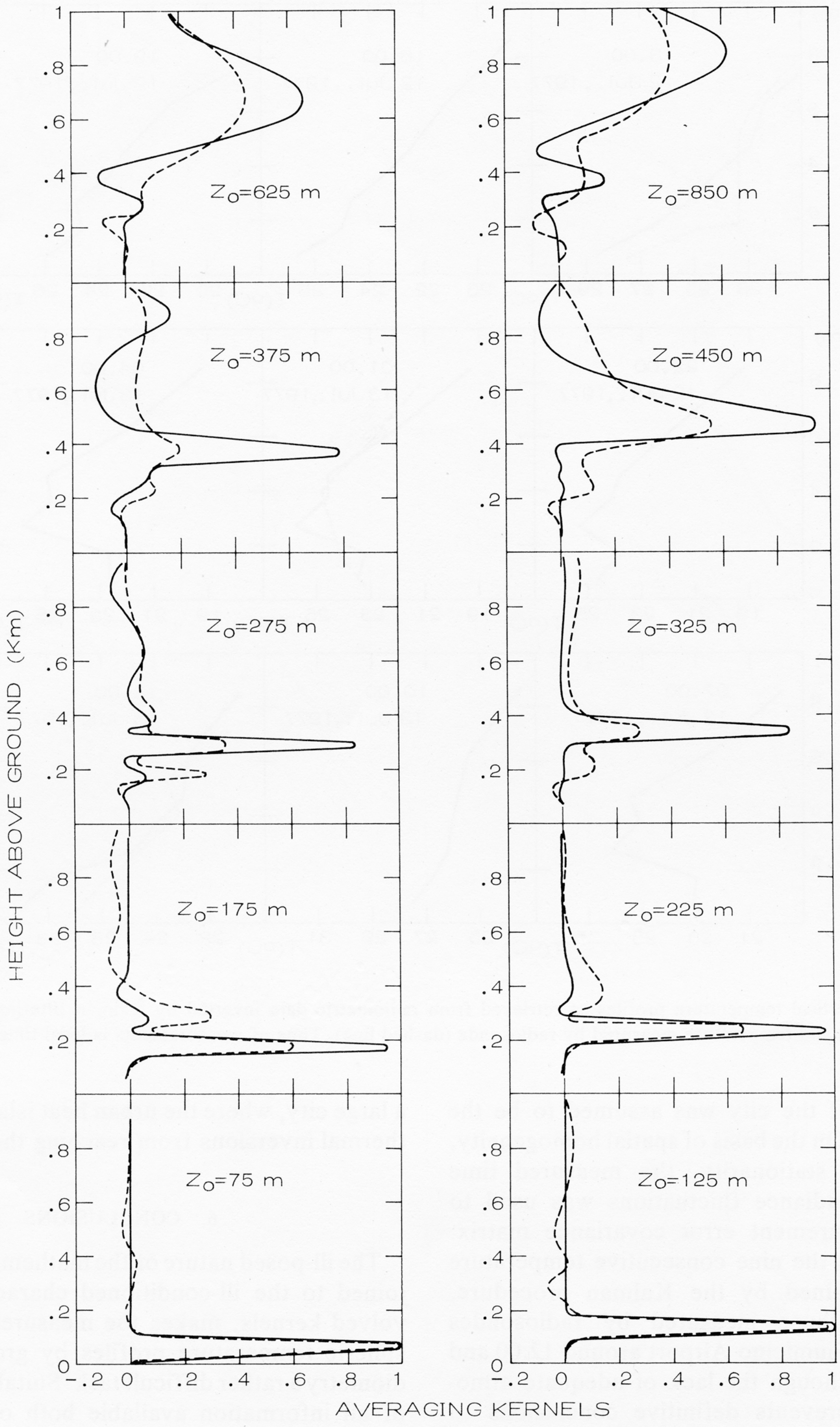

Fig. 2. Summer averaging kernels at various elevations z_0 for the weighting functions of Figure 1 and for measurements at 18 elevation angles between 1° and 35° above the horizon. Solid line refers to Kalman filtering; dashed line refers to Backus-Gilbert inversion.

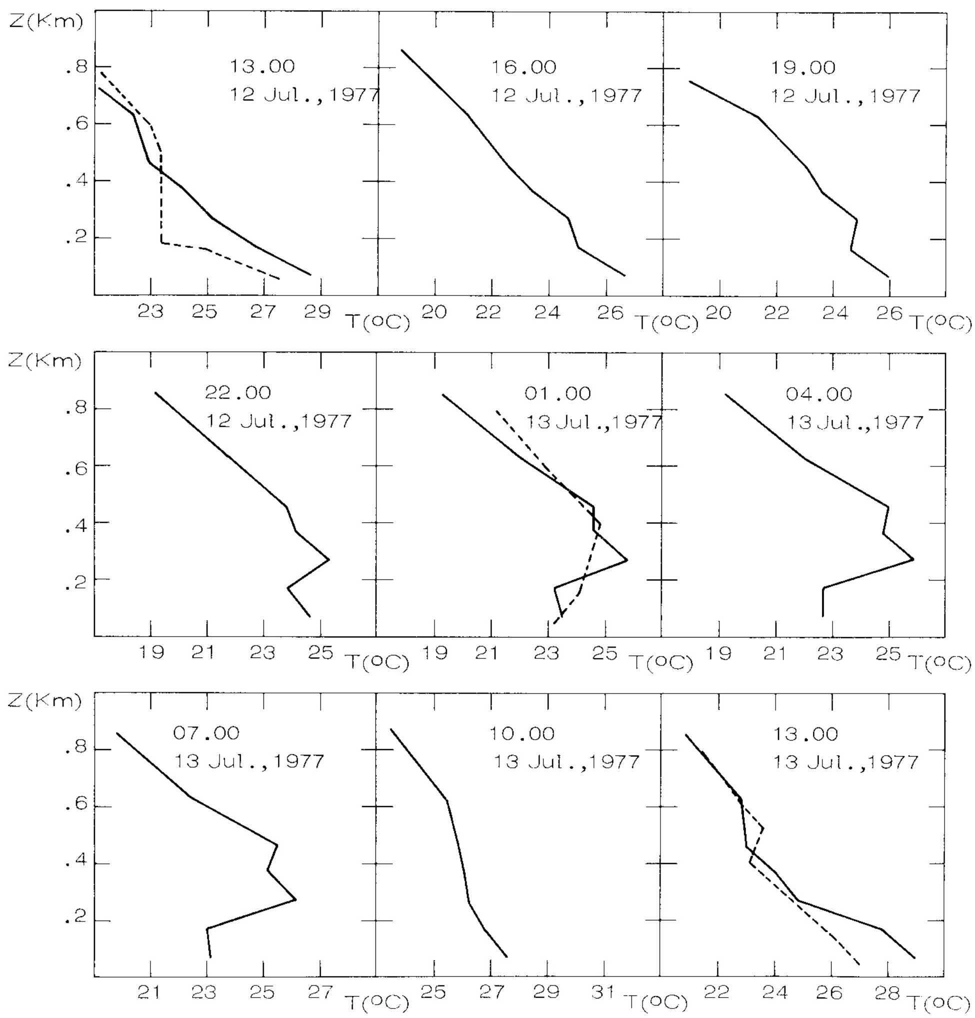

Fig. 3. Vertical temperature profiles as retrieved from radiometric data inverted by Kalman filtering (solid line) and temperature measured by radiosonde (dashed line). Time of measurements is local time.

for the effect of the city was assumed to be the starting profile. On the basis of spatial homogeneity, ergodicity, and stationarity, the measured time covariance of radiance fluctuations was used to build the measurement error covariance matrix. Figure 3 reports the nine consecutive temperature profiles as obtained by the Kalman procedure, together with those measured by radiosondes launched from Fiumicino Airport around 12.00 and 00.00 GMT. Although the lack of adequate atmospheric truth prevents definitive conclusions to be drawn from these results, however, the obtained profiles seem to be fairly representative of the thermal structure of the lower troposphere over a large city, where the urban heat island can prevent thermal inversions from reaching the ground level.

6. CONCLUSIONS

The ill-posed nature of the mathematical problem, joined to the ill-conditioned character of the involved kernels, makes the measurement of atmospheric temperature profiles by ground-based radiometry a rather difficult task. Suitable exploitation of all information available both on the state of the system and on the measurement errors is essential to the significance of the results. The Kalman filtering technique profits by the a priori knowledge about the evolution of the atmospheric thermal state and by the actual accuracy of the radiometric measurements to build a balanced estimation algorithm. By use of this method, some progress has been made in the retrieval of atmospheric temperature profiles from satellite and ground-based radiometric data.

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